CIRCULAR-ARC ROTORS IN REGULAR POLYGONS

MICHAEL GOLDBERG, Bureau of Ordnance, U. S. Navy

1. Introduction. A *rotor* for a polygon is defined here as a convex closed curve which remains tangent to all the sides of the fixed polygon during a complete rotation of the curve. Rotors are a natural generalization of the curves of constant width (also called curves of constant breadth, courbes orbiformes, Gleichdicke) which can rotate between a pair of parallel fixed lines. Since a curve of constant width which is constrained by only one pair of fixed lines is free to slide along these lines, another pair of parallel fixed lines may be added as constraints. Then the curve can rotate in a rhombus which, as a special case, may be a square.

Kakeya, in an oral communication to Fujiwara, showed a method of constructing rotors of regular polygons. Fujiwara investigated rotors more thoroughly, gave a Fourier series for describing them, derived many of their properties and described several examples of them in closed forms [1]. Among these properties he showed that the sides of the polygon must be the sides of a *regular* polygon except for the quadrilateral in which case any rhombus could serve. Hayashi derived a simpler formulation for a class of rotors for all the regular polygons [2].

A circle is a trivial and obvious case of a rotor, although universal, and it will be excluded here from further consideration as a rotor. However, rotors bounded entirely by arcs of circles are of particular interest. In addition to the intrinsic interest of the geometer for these simple rotors, there is the concern of the technician who may have occasion to employ them. The circular-arc rotors are easier to depict and to manufacture. Circular-arc rotors for the square were considered by Euler in 1778 as part of his investigations of curves of constant width. These are described extensively in a paper by Schilling [4]. A circular-arc rotor for the triangle was described in 1875 by the great kinematician, F. Reuleaux, in his *Theoretische Kinematik*. Fujiwara added an infinite series of regular rotors composed of equal arcs of radius equal to the height of the triangle as well as several irregular rotors of five unequal arcs of the same radius [1 and 3]. A circular-arc rotor for the pentagon is described by Fujiwara [3, pp. 245–246]. It seems that for only the three foregoing polygons ($n=3, 4, 5$) have any circular-arc rotors been described in the previous literature.

It is the purpose of this paper to show the construction of circular-arc rotors for all the regular polygons. The construction employs only the most elementary geometry and it serves, therefore, as a simpler introduction to the subject of rotors than any of the previous ones. Other possibilities for these circular-arc rotors are discussed in the last section of the paper.

2. The envelopes of the sides of a moving polygon. Consider the envelopes of the sides of a polygon which moves so that two of the sides, AB and AC, pass through two fixed points P and Q as shown in Figure 1. Through A draw AF_2

parallel to BD and meeting in F_2 the fixed circle which is the locus of A. The angle F_2AP is supplementary to the angle ABD and, therefore, the arc F_2P is constant and F_2 is a fixed point. Also, the perpendicular F_2G_2 from F_2 on BD is equal to the altitude of the triangle ABD. Therefore, BD always touches a circle the center of which is at F_2. This demonstration of a theorem due to Bobillier is given by Besant [5].

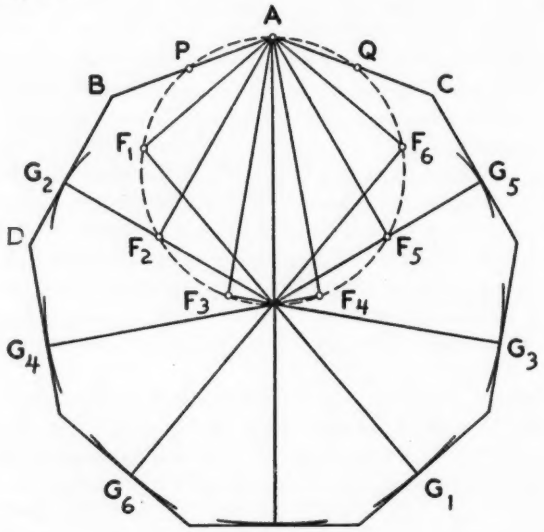

FIG. 1

Similarly, any side of the polygon taken with the sides AB and AC serves as the third side of a moving triangle. The line from A parallel to this third side cuts the circle in a point which is the center of a circular envelope of this moving third side. In general, the centers of the circular envelopes of the moving sides of the polygon are the intersections of the rays from A parallel to the sides of the polygon with the fixed circle. If the polygon is a regular polygon with an odd number of sides, the centers divide the fixed circle into the same number of equal arcs. If the polygon has an even number of sides, then the centers divide the fixed circle into half the number of equal arcs.

3. Location of the arc centers of a circular-arc rotor. Hayashi [2] exhibited a class of rotors which, for the even polygons, have an axis of symmetry, while the rotors for the odd polygons have two axes of symmetry. In the case of a circular-arc rotor, if each radius is reduced by the magnitude of the smallest radius, a rotor is obtained which has one or more sharp corners. When n is even, the rotor is of constant width. If the sharp corner is on the axis of symmetry, then this corner must be the center of the opposite arc which crosses the axis of symmetry. The centers of the arcs adjoining the corner must lie

on the arc opposite the corner. These centers, also, must be corners of the rotor. The centers of the arcs adjoining these corners must lie on the lines joining the corners to the ends of the arcs which adjoin the first corner on the axis.

If the rotor is held fixed and the regular polygon is rotated, the envelopes of the sides of the polygon are the boundaries of the rotor. Place the rotor so that its axis of symmetry coincides with a main diagonal of an even polygon. Then a pair of symmetric corners must lie on two adjacent sides of the polygon. As the polygon is moved either way from this position, the envelopes of the sides of the polygon are circular arcs whose centers lie on a circle through these corners and the vertex of the polygon between them as was demonstrated in Section 2. For the odd polygons, all the centers lie on either of two equal circles. For the even polygons, all the centers lie on either of two unequal circles. Call these circles the generating circles of the rotor.

4. Notation. It will be convenient to employ the following notation.

$n \equiv$ the number of sides of the given regular polygon (n-gon).

$n \equiv 2m$ when the number of sides of the given n-gon is even.

$\alpha \equiv \pi/n \equiv \pi/2m$.

$w \equiv$ the diameter of the inscribed circle of the given regular n-gon. Also, this is the width of a rotor of the given n-gon when n is even.

$4\theta \equiv$ the angular measure of the arc on the smaller generating circle intercepted by the larger generating circle.

$r \equiv$ the radius of the smaller generating circle (through the apex).

$R \equiv$ the radius of the larger generating circle.

$c\{t\} \equiv$ the length of the chord subtending an arc of angular measure t on the smaller generating circle of radius r.

$C\{t\} \equiv$ the length of the chord subtending an arc of angular measure t on the larger generating circle of radius R.

$h\{u\} \equiv$ the height of a segment of the inscribed circle of the given n-gon, where $2u\alpha$ is the angular measure of the arc of the segment.

$a\{u\} \equiv$ the altitude of a segment of the given n-gon bounded by u consecutive edges of the n-gon and the diagonal as base.

5. Conditions on the circular-arc rotor for the $2m$-gon. Consider the case for n even. The relative location of the centers is shown in Figure 2. Then, by equating the radii at the ends of a circular arc, a condition on the variables is obtained. When all such conditions are satisfied, the circular arcs form a continuous boundary. The following equations may then be written:

$$(1)\qquad \begin{cases} c\{t\} = 2r \sin t/2, \\ C\{t\} = 2R \sin t/2, \\ h\{u\} = \tfrac{1}{2}w[1 - \cos u\alpha], \\ a\{u\} = \tfrac{1}{2}w(1 - \cos u\alpha/\cos \alpha) \qquad \text{for } u \text{ odd}, \end{cases}$$

$$(2)\qquad R = w \sin \theta/\sin 2\alpha.$$

The following equations are derived from the centers on the larger generating circle:

$$c\{\pi - 4k\alpha + 2\theta\} + C\{\pi - 4\alpha - 4k\alpha + 2\theta\} + h\{2k\} = c\{\pi - 4\alpha - 4k\alpha - 2\theta\} + C\{\pi - 4k\alpha - 2\theta\} + h\{2k+2\}, \tag{3}$$

where

$$k = 0, 1, 2, 3, \cdots, (m-1).$$

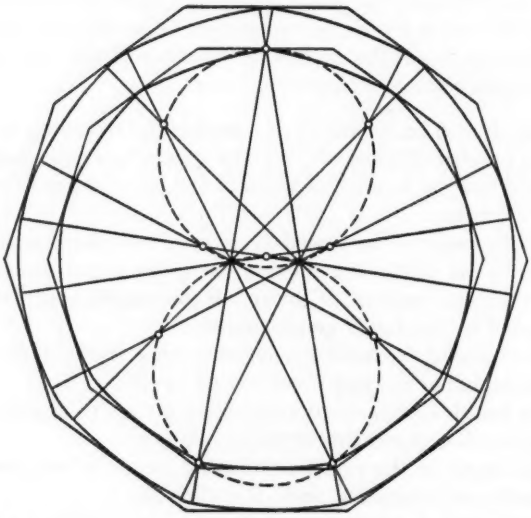

FIG. 2

But from equations (1).

$$c\{\pi - 4k\alpha + 2\theta\} - c\{\pi - 4\alpha - 4k\alpha - 2\theta\} = 2r\left[\sin\left(\frac{\pi}{2} - 2k\alpha + \theta\right) - \sin\left(\frac{\pi}{2} - 2\alpha - 2k\alpha - \theta\right)\right] = 4r \sin(\alpha + \theta) \sin(\alpha + 2k\alpha). \tag{4}$$

From equations (1) and (2),

$$C\{\pi - 4k\alpha - 2\theta\} - C\{\pi - 4\alpha - 4k\alpha + 2\theta\} = 2R\left[\sin\left(\frac{\pi}{2} - 2k\alpha - \theta\right) - \sin\left(\frac{\pi}{2} - 2\alpha - 2k\alpha + \theta\right)\right] = 4w \sin\theta \sin(\alpha - \theta) \sin(\alpha + 2k\alpha)/\sin 2\alpha. \tag{5}$$

From equations (1) again,

$$h\{2k+2\} - h\{2k\} = \tfrac{1}{2}w[\cos 2k\alpha - \cos(2\alpha + 2k\alpha)]. \tag{6}$$

Substituting equations (4), (5) and (6) in equations (3) and dividing by the common factor $\sin(\alpha+2k\alpha)$, we obtain

$$4r \sin(\alpha+\theta) = 4w \sin\theta \sin(\alpha-\theta)/\sin 2\alpha + w \sin\alpha. \tag{7}$$

Note that this equation is independent of k.

The following equations are derived from the centers on the smaller generating circle:

$$\begin{aligned} &c\{\pi - 4k\alpha + 2\theta\} + C\{\pi - 4\alpha - 4k\alpha + 2\theta\} + a\{2k+1\} \\ &\quad = c\{\pi - 4k\alpha - 2\theta\} + C\{\pi + 4\alpha - 4k\alpha - 2\theta\} + a\{2k-1\}, \end{aligned} \tag{8}$$

where

$$k = 0, 1, 2, 3, \cdots, (m-1).$$

But from equations (1),

$$\begin{aligned} &c\{\pi - 4k\alpha + 2\theta\} - c\{\pi - 4k\alpha - 2\theta\} \\ &\quad = 2r\left[\sin\left(\frac{\pi}{2} - 2k\alpha + \theta\right) - \sin\left(\frac{\pi}{2} - 2k\alpha - \theta\right)\right] \\ &\quad = 4r \sin\theta \sin 2k\alpha. \end{aligned} \tag{9}$$

From equations (1) and (2),

$$\begin{aligned} &C\{\pi + 4\alpha - 4k\alpha - 2\theta\} - C\{\pi - 4\alpha - 4k\alpha + 2\theta\} \\ &\quad = 2R\left[\sin\left(\frac{\pi}{2} + 2\alpha - 2k\alpha - \theta\right) - \sin\left(\frac{\pi}{2} - 2\alpha - 2k\alpha + \theta\right)\right] \\ &\quad = 4w \sin\theta \sin(2\alpha - \theta) \sin 2k\alpha/\sin 2\alpha. \end{aligned} \tag{10}$$

From equations (1) again,

$$\begin{aligned} a\{2k+1\} - a\{2k-1\} &= w[\cos(2k\alpha - \alpha) - \cos(2k\alpha + \alpha)]/2\cos\alpha \\ &= w \tan\alpha \sin 2k\alpha. \end{aligned} \tag{11}$$

Substituting equations (9), (10) and (11) in equations (8) and dividing by the common factor $\sin 2k\alpha$, we have

$$\begin{aligned} 4r \sin\theta &= 4w \sin\theta \sin(2\alpha - \theta)/\sin 2\alpha - w \tan\alpha \\ &= [4w \sin\theta \sin(2\alpha - \theta) - 2w \sin^2\alpha]/\sin 2\alpha. \end{aligned} \tag{12}$$

Note that this equation is independent of k.

Equating both values of $(4r \sin 2\alpha)/w$ derived from equations (7) and (12),

$$\frac{4 \sin\theta \sin(\alpha - \theta) + \sin\alpha \sin 2\alpha}{\sin(\alpha+\theta)} = \frac{4 \sin\theta \sin(2\alpha - \theta) - 2 \sin^2\alpha}{\sin\theta}. \tag{13}$$

The following series of expansions and transformations leads to equation (14).

$$2\sin\theta[\cos(\alpha-2\theta)-\cos\alpha]+\sin\alpha\sin 2\alpha\sin\theta$$
$$=2\sin\theta[\cos(\alpha-2\theta)-\cos 3\alpha]-2\sin^2\alpha\sin(\alpha+\theta),$$
$$\tfrac{8}{3}\sin\theta(\cos 3\alpha-\cos\alpha)=2\sin^2\alpha\sin(\alpha+\theta),$$
$$6\sin\theta\cos\alpha\sin^2\alpha=2\sin^2\alpha\sin(\alpha+\theta),$$
$$3\sin\theta\cos\alpha=\sin\alpha\cos\theta+\cos\alpha\sin\theta,$$
$$\tan\theta=\tfrac{1}{3}\tan\alpha. \tag{14}$$

Note that the lines joining the centers of successive arcs of the rotor pass alternately through one and then through the other of the two intersections of the two generating circles. These two intersections may be called focal points of the circular-arc rotor.

The foregoing construction is applicable for $n>7$. For $n=6$, equation (14) does not hold since there are three generating circles instead of two. The radii of the arcs are w and $w/2$. The angular measures of the arcs are 60°, 30° and 15°. For $n=4$, the familiar Euler triangular curve, frequently called the Reuleaux circular triangle, composed of the three equal 60° arcs, may be taken as the corresponding rotor.

6. Conditions on the circular-arc rotor for the odd n-gon. When n is odd, then

$$r=R=a\{n\}\sin\frac{\alpha}{2}\Big/\sin 2\alpha=\frac{w}{2}\left(1+\frac{1}{\cos\alpha}\right)\frac{\sin\frac{\alpha}{2}}{\sin 2\alpha}=\frac{w\cos\frac{\alpha}{2}}{4\cos^2\alpha}$$

and

$$c\{t\}=C\{t\}.$$

In this case, the focal points coincide with centers of arcs. The continuity of the arcs can be verified by the following equations, based on equations (1) and (2):

$$C\{\pi-3\alpha-4k\alpha\}-C\{\pi-5\alpha-4k\alpha\}$$
$$=2R\left[\sin\left(\frac{\pi}{2}-\frac{3\alpha}{2}-2k\alpha\right)-\sin\left(\frac{\pi}{2}-\frac{5\alpha}{2}-2k\alpha\right)\right] \tag{15}$$
$$=4R\sin\frac{\alpha}{2}\sin(2\alpha+2k\alpha).$$

$$C\{\pi-\alpha-4k\alpha\}-C\{\pi-7\alpha-4k\alpha\}$$
$$=2R\left[\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}-2k\alpha\right)-\sin\left(\frac{\pi}{2}-\frac{7\alpha}{2}-2k\alpha\right)\right] \tag{16}$$
$$=4R\sin\frac{3\alpha}{2}\sin(2\alpha+2k\alpha).$$

$$(17)\quad a\{2k+3\} - a\{2k+1\} = w[\cos(2k\alpha+\alpha) - \cos(2k\alpha+3\alpha)]/2\cos\alpha$$
$$= w\sin(2\alpha+2k\alpha)\tan\alpha$$
$$= 8R\cos\alpha\sin\frac{\alpha}{2}\sin(2\alpha+2k\alpha)$$
$$= 4R\left[\sin\frac{3\alpha}{2} - \sin\frac{\alpha}{2}\right]\sin(2\alpha+2k\alpha)$$

Subtracting equations (15) and (17) from (16), we obtain

$$(18)\quad C\{\pi-5\alpha-4k\alpha\} + C\{\pi-\alpha-4k\alpha\} + a\{2k+1\}$$
$$= C\{\pi-3\alpha-4k\alpha\} + C\{\pi-7\alpha-4k\alpha\} + a\{2k+3\}.$$

This equation holds for $k=0, 1, 2, 3, \cdots, (n-1)$. Applying equation (18) to Figure 3 shows that the circular arcs are contiguous.

The foregoing construction is applicable to all polygons of an odd number of sides. In addition, for $n=3$, Fujiwara [5] has described other types of rotors composed of two, four and five arcs.

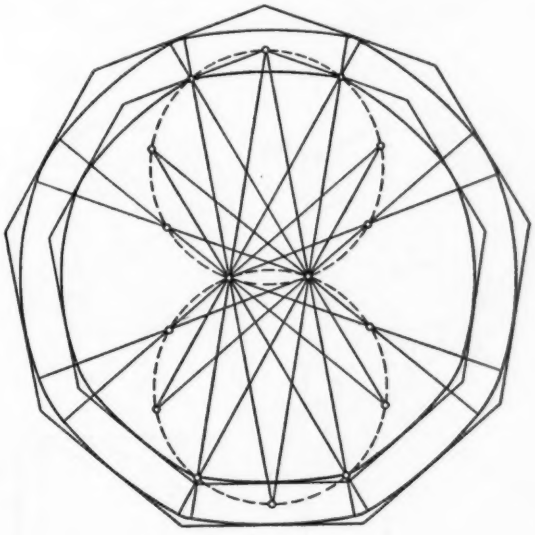

FIG. 3

7. The least number of arcs bounding a circular-arc rotor. A parallel curve to any of the rotors here described is a circular-arc rotor with no corners. (See Figures 2 and 3.) The number of arcs is $2n$ in the case of the even polygons (for n greater than 4) since each center on the generating circles serves as the center

FIG. 4

of two arcs. In the case of the odd polygons, each generating circle carries n center points which serve as centers of single arcs. However, two centers on one generating circle coincide with centers on the other generating circle. Therefore, for odd polygons, the number of arcs bounding the rotor is $2n-2$.

Since at least two generating circles are necessary to produce closure, the rotors described here have the least possible number of arcs if no corners are permitted. When corners are permitted, the number of arcs for even polygons becomes $2n-3$. The equality of the generating circles for the odd polygons permits a rectangular arrangement of four corners resulting in $2n-4$ arcs as the least possible number of circular arcs.

Several examples of rotors with the least number of circular arcs are shown in Figure 4. The dotted circles are the generating circles. The small open circles enclose the centers of the boundary arcs. The small black circles enclose the focal points. Straight lines join the centers of successive arcs and show the sectors determined by the arcs.

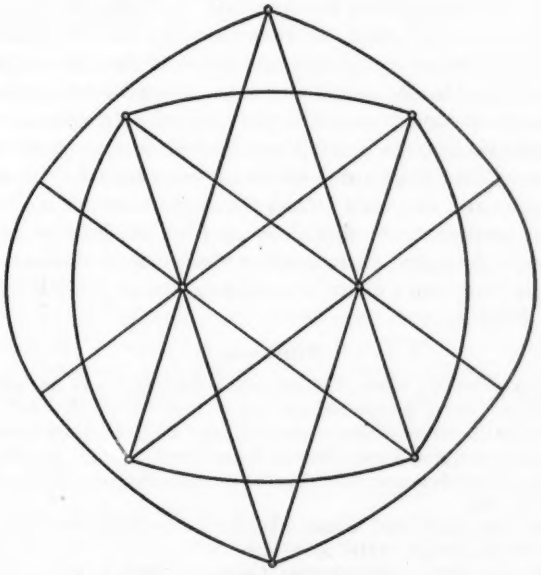

FIG. 5

8. Dual rotors. To every rotor for an odd polygon there corresponds another rotor which plays the rôle of a dual. Just as a parallel rotor is an orthogonal trajectory of the external normals to a given rotor, a dual rotor can be obtained as the orthogonal trajectory of the internal normals if the trajectory is taken at a sufficient distance from the given rotor. The nearby internal trajectories are not simple convex curves. One of these trajectories is a convex rotor with corners;

those on one side are non-convex curves (therefore, they are not rotors), while those on the other side are rotors without corners.

The duals of the circular-arc rotors for the odd regular polygons, described in Section 6, have the same centers and the same number of arcs. However, the duals with corners have only two corners (instead of four) corresponding to the two arcs of maximum radius. The radii of the arcs are always less than the height of the circumscribed polygon. An example of a pair of dual rotors for the pentagon is shown in Figure 5.

The general existence of the dual rotors seems to have been overlooked in the previous literature although one of them was known. The dual composed of two 120° arcs corresponding to the rotor of two 60° arcs, shown in Figure 4, was given by Reuleaux and was discussed by Fujiwara [3].

9. Minimal area of rotors. Fujiwara and Kakeya proposed the problem of determining the rotors of least area for the regular polygons [8]. They showed that for the equilateral triangle the minimal rotor is bounded by two equal 60° circular arcs. It was first shown geometrically by Blaschke [6], and later analytically by Fujiwara [7], that the minimal rotor for the square is bounded by three equal 60° circular arcs. Fujiwara remarked that the method which was successfully employed in the solution of this extremal problem for the triangle and the square was no longer applicable in testing the circular-arc rotor he found for the pentagon even though it could conceivably be the sought minimal rotor. It should be noted that these three rotors are each bounded by the least number of circular arcs and that each includes arcs whose radius is the height of the polygon. These properties are shared by each of the infinite series of rotors, which includes the foregoing three cases, whose constructions are given in this paper. Therefore, they are worthy of consideration as possible solutions of the minimal rotor problem.

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THE LINKING OF MATHEMATICS AND PHYSICS IN UNIVERSITY COURSES

L. ROSENHEAD, University of Liverpool,
C. STRACHAN,* University of Aberdeen

1. Introduction. The past thirty years have seen a great intensification of the application of mathematical methods to the study of natural phenomena. Associated with this there has been a widening and deepening of our understanding of the nature of matter and radiation, and an increase in the power of designers in many branches of technology. The prospects of further technological developments resulting from our increased understanding are very great and many universities propose to equip some of their students with the techniques necessary to enable them to play an increasing part in the new scientific and technological advances. This, of course, means a linking of the instruction given in mathematics and physics, even closer than that which has been obtained in the past, and it also means consequential changes in some of the instruction given within faculties of engineering.

2. Mathematics and physics. The formal linking of mathematics and physics in English universities is entering upon an interesting phase. In the past there have been few or no attempts by departments of physics to teach their "own mathematics," or by departments of engineering to insist upon giving instruction in mathematics within the walls of their own faculties. Between professors and teachers of all departments and faculties, there has been good-natured disagreement and also much friendly agreement on educational matters. This has led to a cooperative outlook and a willingness to adapt and modify existing courses of instruction.

There are, of course, difficulties inherent in the attempt to cooperate, and to some extent these are reflected in the names which are used to denote various branches of instruction; examples of these are pure mathematics, applied mathematics, mathematical physics, theoretical physics, chemical physics, physical chemistry, metal physics, physics, and so on. Of these, "pure mathematics" alone seems to have retained its time-honored significance, though the content of the instruction given under this heading has changed considerably. "Applied mathematics" used to mean "applications of mathematics," and involved drastic idealization of much of the physical conditions of reality. Surprisingly accurate and surprisingly inaccurate predictions resulted from this, and the uncertainty of the validity of the conclusions arising from the combination led to applied mathematics being regarded with some suspicion.

With the passage of time, however, as mathematicians applied their techniques to problems in which the physical assumptions bore a greater resemblance to reality than previously, and as engineers and physicists became more and more expert in the techniques of pure mathematics, a reasonably happy com-

* Formerly of the University of Liverpool.

promise was effected. "Applied mathematics" came to be known as covering in the main "engineering mathematics"; that is, it included mechanics, hydrodynamics, aerodynamics, elasticity, magnetism and electricity, and so on. Astronomy and gravitational theory do not fit in with this rough sub-division, but the subjects were loosely grouped under applied mathematics. "Physics" was understood to cover, in the main, experimental investigations, but it came to use more and more the terms and techniques of pure mathematics in order to explain its outlook and to convey its results. "Mathematical physics" was understood to mean a linking of mathematics and physics, but with even greater emphasis on the methods of pure mathematics. "Theoretical physics" was accepted as a variant of mathematical physics, in which mathematics and physics were combined with assumptions whose foundations might be rather difficult to formulate in precise terms. These sub-divisions are, of course, extremely rough and there is considerable overlap between them. Change and development continue, and many of us, as members of universities, are trying to lay the foundations for developments likely to bear fruit during the next quarter of a century. Much thought is therefore being given to the proper place of mathematics and physics in the general scheme. We are concerned with mathematical physics, both as a part of a general education, and as an aid to detailed understanding of physical science and technological advance. Education in mathematical physics ought to, and generally does, begin at school.

3. Instruction in mathematics and physics. In many schools pure mathematics is taught as a series of interesting and stimulating puzzles, often necessitating the important mental pastime of deciding what is, and what is not, relevant. This is usually done without an attempt to give a comprehensive picture of what mathematics is or what it purports to be. Then, often under the title of applied mathematics, the elements of physics are introduced and mathematical methods are applied to what is simply a new field of activity introduced by means of bewildering, and often ill-considered, definitions. These may well have the effect, later, of repelling many keen young minds from pure mathematics, mathematical engineering, or mathematical physics. At an early stage the pupil develops crude ideas of physical quantities such as force, mass, and so on, and it is generally only in the universities that an attempt is made to reduce these ideas to some semblance of order. This attempt should, we feel, be made at a much earlier stage. Teachers might take classes into their confidence when pupils are at quite an early age, say even at eleven years, and talk to them on the genesis of mathematics. First of all, it should be presented as essentially a practical subject. Its historical development should be described with special reference to its relation to the problems of primitive societies, such as the counting of possessions and the measuring of land. Then might be explained the fumbling, halting investigations which formulated the properties of numbers, irrespective of the particular objects to which the numbers were attached. At a slightly later stage, at the beginning of the study of mathematical engineering

and mathematical physics, the ideas and definitions of mechanics might be introduced, including the use of machines and the mathematical problems arising in the precision of their design, the effects of matter in motion, the problems which the engineer has to solve, and so on. These might all serve to arouse the interest of the pupil. Definitions might be regarded as statements formulated and gradually made more precise, possibly by the pupils themselves; their display on a large, accessible blackboard would perhaps keep them in mind. Above all, certain types of "definitions" must be avoided, at any rate as the first words on the particular topics to which they refer. The following, for example, are the first words offered by well-known books on the topics which they purport to describe: "The moment of a force about a point is the product of the force and the perpendicular drawn from the point on the line of action of the force," and again "Matter is that which has mass," and still further, "The mass of a body is the quantity of matter in the body." This method of approach must be shunned!

It is very unwise to attempt to achieve apparent academic success by brevity of discussion and of definition when it is accompanied by neglect of the all-important scientific background required for the definition or discussion. Indeed, while too much detail may be confusing, and a compromise between amount of detail and reasonable clarity must be sought, it has to be remembered that too much willingness to abstract and isolate physical phenomena may well develop an attitude toward general experience which may make its possessor a menace to the community in which he lives.

Even at this early stage the method of mathematical physics should be emphasized—the sequence of experiment, thought about the circumstances which are relevant to the phenomenon studied, the tentative (and not always reliable) rejection of circumstances which seem not to be really significant and whose inclusion impedes mathematical formulation, mathematical formulation and solution, final comparison with experiment, and, very often, the decision to repeat the whole sequence with different assumptions about relevance.

The qualities and defects of the instruction in mathematics and physics given in schools are carried over to the universities. In many universities in this country there has been an exaggerated division between those studying the mathematical and experimental aspects of physical science. The enormous development of the field of study has, of course, been the main cause of this. Formerly there was a general tradition of "natural philosophy" in teaching. This involved detailed acquaintance with both experimental and mathematical physics as complementary aspects of one subject. Nowadays it is difficult in an undergraduate course to develop both aspects to provide sufficiently powerful equipment. The policy has been (1) to combine training in mathematical physics, under the name and limitations of applied mathematics, with a full course in pure mathematics including branches of pure mathematics of very specialized interest although admittedly of great aesthetic value, such as branches of pure geometry, parts of the theory of numbers, and so on, or (2) to combine training in experimental physics with an inadequate training in mathe-

matical physics. Of these (1) means limitation to a very narrow field of study and (2) involves giving mathematical training of an almost entirely "set" nature with little or no problem work.

It is clear that the mathematical physicist, in an undergraduate course, cannot be expected to become really proficient in the modern technique of elaborate experiment, but he has a right to demand that his mathematical training will give him reasonable equipment in the mathematical techniques which he really needs, in the most economical manner. He must have an understanding of mathematical method and, at the same time, a clear grasp of physical principles. This must not suggest that the mathematical physicist is an unfortunate hybrid whose conclusions are likely to be unsatisfactory or inadequate. The union of mathematical technique with appropriate understanding and knowledge of physical principles often produces results of as great significance as does the wider equipment in either field alone. For example, in many types of physical theory, and certainly in atomic and nuclear physics, it is not possible to have simple and easily interpreted fundamental experiments on which to build a theory. The experimental verifications of the results to be expected from any physical "model" have to be made at a considerable experimental and mathematical distance from the fundamental mathematical and physical hypotheses. Then mathematical intuition may play the decisive part in formulating quantitative hypotheses. Examples abound in the use of mathematical properties of invariance under Lorentz and other transformations in the theories of elementary particles, the symmetrical and anti-symmetrical wave-functions of quantum mechanics, and even in the displacement current of classical electrodynamics. Indeed, it is wrong perhaps to speak here of mathematical intuition. The mathematical principles used are often simply the precise formulation of conceptions of space, time, and so on, which we have acquired physically, not in a brief laboratory experiment, but by generations of slow endeavor.

In the attack on any problem in natural philosophy it is important that the mathematical physicist should use a technique of adequate power. That uneconomical training should limit him to use crude mathematical methods when more powerful methods are readily available and assimilable with a little effort, is no less culpable than that the experimenter should be limited to antiquated and unreliable techniques in experiment.

Mathematical physicists will hope for the full collaboration of the pure mathematician in the development and teaching of satisfactory and powerful techniques. In much of modern, and of classical, mathematical physics the really relevant and powerful mathematical methods have sufficient intrinsic mathematical elegance to provide that attraction of interest which will reward the pure mathematician for his help and will in itself attract the student. At present, in this country, much could be done for the mathematical physicist who is anxious to become proficient in methods of algebra and group-theory in order to attack problems in molecular, atomic and nuclear physics; this is merely one example.

4. A glance at the future. What then may we hope for in the future? The need for subdivision of natural philosophy into separate schools of experimental and mathematical physics will remain, although the two schools must to a certain extent overlap in their studies. For perhaps three of the usual four years of an undergraduate honors course, it is advisable for the mathematical physicist to take adequate courses in experimental and descriptive physics while, at the same time, taking intensive and specially planned courses in pure mathematics and mathematical physics. On account of the great width of the subject there will certainly have to be a careful choice of the branches of physics which are to be studied; it is probably wise to limit the width of the field of study by demanding an effective and precise quantitative understanding of what is studied, assuming that the function of the mathematical physicist is to offer reasonably precise opinions on his field of study rather than nebulous opinions on a very wide field. In subjects of basic importance such as classical mechanics and electrodynamics the aim should be to give a firm and precise grasp of the fundamental physical principles and the most important mathematical techniques; it is not necessary to discuss almost every type of problem which has ever been of importance in these subjects. The need for cooperation and even self-denial from pure mathematicians has been mentioned before; very often they will have to omit topics of great importance for them and to prove or, occasionally in difficult matters, even merely to make plausible, certain theorems under much more restrictive conditions than are really necessary. That their reward will be a rich one, in interest and respect, is quite certain.

In the last year of the course practically all of the student's time will have to be given to the study of experimental results and the development of mathematical and mathematical-physical techniques. It is advisable that he should see certain important types of experimental arrangement and perhaps take some part in the performance of a few carefully selected experiments. However, he will not in general be able to spare time for detailed study of experimental techniques.

It is perhaps unavoidable that, at this stage, there should be a considerable segregation of students of various branches such as molecular, atomic, and nuclear physics, hydro- and aerodynamics, and so on. One can only hope that it will not be too extreme.

5. Present courses of study. In the University of Liverpool we have made an attempt to formulate suitable and practicable courses leading to a degree with honors in mathematical physics. Quotations from parts of our scheme follow. The actual syllabuses given refer to some of the basic subjects in pure mathematics and mathematical physics. On account of the absence of so many of our physicists on war work, the courses in descriptive and experimental physics could not be adequately discussed and detailed syllabuses for these are therefore not included.

EXAMINATIONS AND COURSES OF STUDY

The period of study normally required for the degree of B.Sc. with Honors in Mathematical Physics will be four years. In the first three of these the courses will be: *Intermediate year*, Introductory Pure Mathematics, Applied Mathematics, Physics; *Part I year*, Courses in Pure Mathematics, Applied Mathematics and Physics; *Part II year*, Courses in Pure Mathematics, Applied Mathematics and Physics. Detailed syllabuses for these, and for some of the Honors courses, are given below, together with notes on admission to the courses.

In the Honors Examination, at the end of the fourth year of study, students will be expected to offer papers in three groups. Those of Groups I and II, seven in number, will be compulsory and will cover certain basic equipment. Those of Group III, two in number, will allow choice in accordance with the candidate's interests. They will be chosen by the candidate in consultation with the Heads of the Departments of Mathematics and Physics.

For these students, there will be no Honors Examination in Experimental Physics. They should, however, be able to spend some time in a Physics Laboratory in the honors year, without unduly overloading their courses, as they will have taken suitable courses and examinations in Experimental Physics in the earlier years of study.

The papers in Group I are:
Methods of Mathematical Physics (one paper),
Statistics and Numerical methods (one paper),
Classical Mechanics and Vector Field Theory (one paper).

The papers in Group II are:
Quantum Theory of Matter and Radiation, Molecular, Atomic and Nuclear Physics (three papers),
Essay paper on the Principles of Physics (one paper).

The papers in Group III are:
Advanced methods in Mathematical Physics including the Theory of Groups, in relation to problems of Mathematical Physics (one paper),
Thermodynamics and Statistical Mechanics (one paper),
Relativity and special problems in Electromagnetic Theory, e.g. Optics, etc. (one paper).

The subjects to be studied in any session will be announced at the beginning of that session.

PROPOSED SYLLABUSES

The following are suggested syllabuses for some of the lecture courses; they will be modified in the light of experience.

INTRODUCTORY COURSES (similar to those given at present):

PART I COURSES:

(A) Pure Mathematics Course (5 lectures per week).
Functions, graphs, continuity. Differential calculus. Expansions in power series. Orders of magnitude. Integration of elementary functions. Solution of ordinary differential equations, including solution in series. Fourier series. Beginning of a calculus of many variables. Functions of many variables. Maxima and Minima, Taylor Series, Euler's Theorem and repeated Cartesian and Polar Integrals. $B(p, q)$ and $\Gamma(x)$. Complex numbers. Determinants (excluding multiplication). Linear equations. Plane differential geometry. Elements of theory of equations. Language of matrices. Quadratic forms. Linear geometry in two and three dimensions. Vectors (brief revision from Applied Mathematics). Elements of conics and quadrics.
(B) Applied Mathematics Course (4 lectures per week).
Theory of vectors with illustrations up to and including scalar and vector products and the two triple products; differentiation of a vector function of a single variable, with illustrations. Foundations of mechanics (elementary). Statics: coplanar forces, friction, theory of mass systems, and stresses in beams and flexure of beams. Dynamics: kinematics of a particle and kinematics of a rigid body for the simpler types of motion. Dynamics of a particle: simple harmonic motion, damped oscillations, forced oscillations, projectiles, central orbits (inverse square law mainly). Work and energy. Kinematics and dynamics for two particles: normal modes for two particles (elementary). Kinematics and dynamics of a system of particles and of a rigid body: principles of momentum and energy, illustrations mainly from two-dimensional motion. Mechanics of continua. Hydrostatics: perfect fluid, resultant thrust, center of pressure, floating bodies. Gases. Elementary elasticity.
(C) Physics Course (5 lectures and 5 hours laboratory work per week).
Similar to the present Part I Course: 2 lectures per week on properties of matter and heat, 3 lectures per week on electricity and magnetism and elementary electronics. 5 hours Experimental Physics.

PART II COURSES:

(A) Pure Mathematics Course (3 lectures per week).
Descriptive account of bounds, limits, convergence of easy real and complex series, especially radius of convergence of power series. Theory of integration. Infinite integrals. Multiple integrals. Green, Gauss, and Stokes theorems. Partial differential equations of Physics. Normal solutions, boundary conditions, first account of eigen-functions and orthogonal series. Study of P_n, J_n, *etc.* Determinants, elementary matrices. Language of groups. Introduction to differential geometry of curves and surfaces. Introduction to calculus of variations. Transformations, Jacobians. Functions of complex variable. Easy contour integration.

(B) Applied Mathematics Course (3 lectures per week).
Mechanics: Moments of inertia and principal axes of inertia for three-dimensional distributions. More general motion of a rigid body. Lagrange's equations. Gyrostatic problems. Normal coördinates. Principle of Least Action (elementary). Electricity and Magnetism: with accompanying theory of vector fields. The course will cover the fundamental parts of the subject as far as and including Maxwell's equations. General: Simple examples of diffusion and wave phenomena. An introduction to the special theory of relativity.
(C) Physics Course (4 lectures and 5 hours laboratory work per week).
3 lectures per week on Optics and Atomic Physics and 1 per week on Electronics. 5 hours Experimental Physics.

HONORS COURSES:

(A) Methods of Mathematical Physics (60 lectures per session).
Complex function theory. Contour integration and conformal representation. Theory of linear differential equations leading to Sturm-Liouville theory. Asymptotic Expansions. Fourier integrals. Differential geometry. Calculus of variations. Tensor calculus (introduction). Optional topics chosen from Lebesgue integration, Fourier series, and General Linear Algebras.
(B) Classical Mechanics and Vector Field Theory (30 lectures per session).
Three dimensional statics (briefly). Three dimensional kinematics. General Dynamics including Lagrange's equations (continued). Theory of vibrations. Hamilton's equations, Hamilton-Jacobi equation. Variational principles. Transformation theory of dynamics. Maxwell's equations and electromagnetic waves. Scalar and vector potentials: retarded and advanced potentials. Electron theory: Lorentz force, *etc.* Principles of energy and momentum. The special theory of relativity. Equations of continuity, motion, and pressure of non-viscous fluids. Rotational and irrotational motion. Velocity potential. Stream function. Complex potentials. Images. Vortices.
(C) Statistics and Numerical Methods. (30 lectures per session and time for practical numerical work.)
Statistics with special reference to the analysis of experimental results. Interpolation. Integration. Solution of algebraic, transcendental and differential equations. Harmonic analysis.

ADMISSION TO COURSES

The admission of students to Part I courses, Part II courses, and honors courses would be decided by the Professors of Pure Mathematics, Applied Mathematics and Physics. They would adopt suitable criteria; for example, entry into Part I courses would be given by a suitable mark at the Introductory Standard or in the Higher School Certificate in the subjects concerned. It might for a few years be advisable, as at present, to allow very good qualifications in H.S.C. (or other equivalent examination) in Pure Mathematics and Physics to

compensate for somewhat inadequate qualifications in Applied Mathematics. Such a deficiency could be helped by the provision of a tutorial class in the second year of study.

In addition, an effort should be made to help students who decide rather late in their courses, say at the end of the Part I year, that they wish to qualify as mathematical physicists. They might be admitted to the necessary courses, provided that they have shown suitable ability, and if they rectify the omissions in their earlier courses by private reading.

ON INDECOMPOSABLE POLYHEDRA

F. BAGEMIHL, University of Rochester

1. Introduction. We shall prove the following

THEOREM. *If n is an integer not less than 6, then there exists a polyhedron, π_n, with n vertices and the following properties*:

(I) *π_n is simple, and every one of its faces is a triangle.*

(II) *If τ is a tetrahedron, each of whose vertices is a vertex of π_n, then not every interior point of τ is an interior point of π_n.*

(III) *Every open segment whose endpoints are vertices of π_n, but which is not an edge of π_n, lies wholly exterior to π_n.*

It is well known that every *convex* polyhedron can be decomposed into a set of tetrahedra whose vertices are all vertices of the given polyhedron [1, p. 280 or 2, p. 57]; and *every* simple polygon can be decomposed into a set of triangles whose vertices are all vertices of the given polygon [1, p. 246 or 2, p. 46]. Lennes, however, proved [2, p. 55] the existence of indecomposable polyhedra by constructing a polyhedron which has properties (I) and (II). His polyhedron, which possesses seven vertices, does not satisfy (III). Schönhardt [3] subsequently gave an example of a polyhedron having six vertices and all three of the above properties. He showed, moreover, that there is no indecomposable polyhedron with less than six vertices.

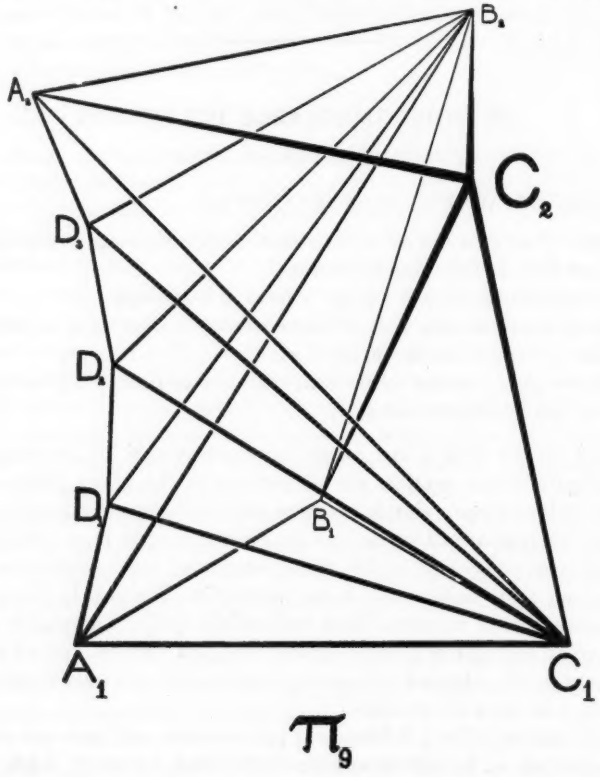
We shall take π_6 to be a Schönhardt polyhedron, and base our construction of π_n, for $n>6$, on π_6. It will be apparent first that, for every $n\geqq 6$, π_n satisfies (I) and the following condition:

(IV) *Every triangle whose sides are edges of π_n is a face of π_n.*

Then we shall demonstrate that (III) holds. Finally, (II) follows from (I), (III), and (IV). For suppose that, on the contrary, there existed a tetrahedron, τ, with its vertices and interior points consisting exclusively of vertices and interior points, respectively, of π_n. Then, because of (III), every edge of τ would be an edge of π_n, otherwise some interior point of τ would be exterior

to π_n; by (IV), every face of τ would be a face of π_n; and, according to (I), π_n is simple, so that π_n would have to be identical with τ. But this would contradict our assumption that $n \geqq 6$.

2. Description of π_6. Let A_1, B_1, C_1 be the vertices of an equilateral triangle, each of whose sides has length 1; and let A_2, B_2, C_2 be the vertices of the triangle obtainable from triangle $A_1B_1C_1$ by first rotating the latter about its center, and in its plane, through 30° in the direction $A_1B_1C_1$, and then translating it one unit in the direction perpendicular to the plane $A_1B_1C_1$, as in the figure, where



A_2, B_2, C_2 correspond, respectively, to A_1, B_1, C_1 under this transformation. Then π_6 consists of

6 vertices: A_1, B_1, C_1; A_2, B_2, C_2;

12 edges: A_1B_1, B_1C_1, C_1A_1; A_2B_2, B_2C_2, C_2A_2; A_1A_2, B_1B_2, C_1C_2; A_1B_2, B_1C_2, C_1A_2;

8 triangular faces: $A_1B_1C_1$, $A_2B_2C_2$; $A_1A_2C_1$, $B_1B_2A_1$, $C_1C_2B_1$; $A_1A_2B_2$, $B_1B_2C_2$, $C_1C_2A_2$.

It is evident that π_6 possesses properties (I) and (IV); and (III) also holds, because the only (open) segments whose endpoints are vertices of π_6, but which are not edges of π_6, are A_1C_2, B_1A_2, C_1B_2, and these are obviously wholly exterior to π_6.

3. Construction of π_n, $n>6$. Let $n=6+k$. In the interior of π_6, take an open circular arc $\overset{\frown}{A_1A_2}$, with endpoints A_1, A_2, and with a radius so large, that every point of $\overset{\frown}{A_1A_2}$ is on the same side of the plane $C_1A_2C_2$ as A_1, and on the same side of the plane $B_1A_1B_2$ as A_2. (See the figure.) On $\overset{\frown}{A_1A_2}$, choose k distinct points, $D_1, D_2, \cdots, D_k$, in the order $A_1D_1D_2 \cdots D_{k-1}D_kA_2$. Then π_n shall consist of

n vertices: $A_1, B_1, C_1; A_2, B_2, C_2; D_1, D_2, \cdots, D_k$;

$3k+12$ edges: $A_1B_1, B_1C_1, C_1A_1; A_2B_2, B_2C_2, C_2A_2; B_1B_2, C_1C_2; A_1B_2, B_1C_2, C_1A_2; A_1D_1, D_1D_2, D_2D_3, \cdots, D_{k-1}D_k, D_kA_2; C_1D_1, C_1D_2, \cdots, C_1D_{k-1}, C_1D_k; B_2D_1, B_2D_2, \cdots, B_2D_{k-1}, B_2D_k$;

$2k+8$ triangular faces: $A_1B_1C_1, A_2B_2C_2; B_1C_2B_2, C_1B_1C_2, A_1B_2B_1, A_2C_1C_2; C_1A_1D_1, C_1D_1D_2, C_1D_2D_3, \cdots, C_1D_{k-1}D_k, C_1D_kA_2; B_2A_1D_1, B_2D_1D_2, B_2D_2D_3, \cdots, B_2D_{k-1}D_k, B_2D_kA_2$.

Obviously π_n satisfies (I); and it is easy to see that (IV) also holds, by simply comparing the various triangles in question, with the list of faces of π_n.

The open segments whose endpoints are vertices of π_n, but which are not edges of π_n, are

$A_1C_2, B_1A_2, C_1B_2; D_1C_2, D_2C_2, \cdots, D_kC_2; D_1B_1, D_2B_1, \cdots, D_kB_1$; and all segments connecting pairs of nonadjacent vertices of π_n on $\overset{\frown}{A_1A_2}$, its endpoints included. It is clear that every segment of this last class is wholly outside π_n, because $\overset{\frown}{A_1A_2}$ is a circular arc. A_1C_2, B_1A_2, C_1B_2 were seen to be completely exterior to π_6; and since $\overset{\frown}{A_1A_2}$ was chosen to lie *within* π_6, these segments are also entirely exterior to π_n. Every one of the segments $D_1C_2, D_2C_2, \cdots, D_kC_2$ is wholly outside π_n. For, each intersects the (open) triangle $A_1A_2C_1$, because $\overset{\frown}{A_1A_2}$ was taken to lie inside π_6 and, at the same time, on the same side of the plane $C_1A_2C_2$ as A_1. Consequently, it is obvious that the only faces of π_n which any of these segments could possibly intersect, are the faces $C_1A_1D_1, C_1D_1D_2, \cdots, C_1D_{k-1}D_k, C_1D_kA_2$. But, since $\overset{\frown}{A_1A_2}$ is a *circular* arc, any open segment joining a point of this arc to a point of the open triangle $A_1A_2C_1$ cannot contain a point of any one of the faces in question. An analogous argument shows that every one of the segments $D_1B_1, D_2B_1, \cdots, D_{k-1}B_1, D_kB_1$ is completely exterior to π_n. Hence, π_n satisfies (III).

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COLLEGE ENTRANCE REQUIREMENTS IN MATHEMATICS*

A. J. KEMPNER, University of Colorado

1. The present situation. In his retiring Presidential Address, "The Scholar in a Scientific World,"** Professor MacDuffee characterized many of the outstanding difficulties confronting our educational system in so masterly a manner that I am tempted to ask you to consider my remarks largely as footnotes and comments to his inspiring talk.

In 1946, the University of Colorado "liberalized" its entrance requirements. The new pattern will be painfully familiar to many among you. Three units of English are required, besides nine units of "academic" subjects, plus three free electives. The academic subjects may not be selected arbitrarily, but students are allowed to enter the University without any high school work in any chosen one of the four large fields: foreign languages, mathematics, social sciences, physical sciences. It is even possible for students to enter without any work in any two of these four groups.

In mathematics the situation is aggravated by the fact that a student who wishes to offer mathematics for entrance may substitute "general mathematics" and "high school arithmetic" for algebra and geometry. In this particular point the proponents of the revision have been remarkably inconsistent. When these substitute courses were introduced in the Colorado high schools, "general mathematics," as the term is understood in our part of the country, was planned as a course intended specifically for students who were admittedly incapable of carrying the standard algebra and geometry courses, and who did not intend to go on to college. This "general mathematics" must not be confused with "unified mathematics," for which a very strong case can be made out. You would have to see some of these general mathematics textbooks to realize what is being done to mathematics in the secondary system; there is no emphasis on proofs, no logical derivations of any but trivial theorems, no logical sequence. The theory of quadratic equations may be compressed into a single page, the formula for the solution of quadratic equations may be given without any derivation, and so on. Emphasis is placed entirely on memorizing formulae and applying them mechanically. I am not arguing against the possible usefulness of such courses for the class of students for which they were supposedly intended; but the fact that these same courses are now admitted as satisfying entrance requirements, cannot be condemned too severely. This point, as much as the elimination of mathematics as a required subject, has unified the Colorado high school teachers in their determined fight against the new rules.

To this group I can speak only of the mathematical aspects of the program. But its real importance goes far beyond the confines of mathematics. It repre-

* One paper of a Symposium, "College entrance requirements in mathematics," held at the thirty-first annual meeting of the Association, Athens, Georgia, January 1, 1948. The second paper will appear in the October issue.

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sents one small link in the long chain of gradual lowering of standards which is relentlessly pushing us into a unique position of educational inferiority among the civilized countries of the world. It would be very easy to substantiate this statement. However, for lack of space and time, I mention only a few isolated items which will illustrate the point. A couple of years ago a student from Irak entered our university. To place him in mathematics, I gave him our mathematics entrance test. After looking it over, he asked me whether I had not made a mistake. He said that he wanted to enter the university, not high school. I explained as well as I could, but he was a sorely puzzled individual. A graduate of the University who had spent a summer in Mexico City attending the University there, told me that he could not muster courage to tell his fellow students that he held a degree from an American University. He said, "I was too ashamed of my degree." What he had to say about his American education cannot be printed. When I was in Vienna last and was asked by the members of the mathematical faculty to tell them what was expected of entering students I found it quite impossible to explain the situation to them. It amounted to "shouting across seas of misunderstanding."

2. Placing the responsibility for this steady lowering of standards. The colleges of education and the administrative circles in the secondary school system, which have been in charge of the development of secondary education, must in the nature of things assume primary responsibility. Child centered programs, the various theories of progressive education, education for life, and so forth, one after another, each has been tried and found insufficient, but each one has left the core of our educational system weaker than it had been. Never, in the more than a third of a century that I have taught in this country, has there been in any case even a temporary strengthening, as far as adequate preparation for college is concerned.

However, it would be quite unfair to place the whole blame on these circles. The college teachers of mathematics and of all branches of learning, and groups such as our own Association, cannot evade a large share of responsibility. We have been willing to let the professional educators do all of the hard and dirty work, and, as always, "He who pays the piper, calls the tune." In saying this, I am in no way forgetful of the highly important work done by many joint commissions over a long range of years, in which the Association has played an important rôle, nor am I unmindful of the unselfish and devoted efforts of many individuals, some still alive, others no longer among us. Anyone who has followed developments attentively, will know what I mean.

3. The arguments employed by colleges of education and by administrators. In essence, the arguments advanced boil down to the following:

(1) *Less than one-fifth of the high school population goes on to college. It is unreasonable to allow the tail to wag the dog, and to let the requirements of a small minority of students dictate standards for the much larger group.*

One's first reaction to this argument is divided between a sense of dismay at a philosophy of education which tends to sacrifice deliberately the training

of our best students to the limit of their capacity, and an admission that the statement does not sound unreasonable on the surface. We are dealing with a very real and very difficult problem. But the argument is really completely unsound. To take care of the situation, we have already available double track plans, possible consolidation of school districts, extension and correspondence courses by state universities, and so on. A radical improvement will of course have to await a thorough overhauling of our whole educational system and theory.

But the following consideration should be fundamental: For the four-fifths who do not go on to college, the high school education is terminal. Granted that a high percentage of high school students cannot profit by abstract training, it may be estimated that more than one-half of them can very well do so, and they, even more than those going to college, stand in need of the most thorough training they can absorb. If we ignore this, as we are doing to an alarming degree, we are gambling with the whole future of our civilization and with the very future of our country.

One of my colleagues at the University of Colorado told me in all seriousness: "If a student can read and understand what he has read; if he can work arithmetic problems correctly (not algebra), and if he has a high I.Q., I am willing to admit him to the University." Such ideas are widespread, although not frequently expressed quite so frankly. It is small consolation that we all too often do not even get that!

Last year in a Denver meeting of Colorado high school principals and superintendents one member stated that in his high school the only fixed prerequisite for graduation is one-half unit English, and propounded the thesis that the whole educational system from the grades through the university forms one unit, and just as the high schools have to take all students fed into them from the grades, so the colleges should be required to accept sight unseen all high school graduates, without having anything to say concerning standards. There was in the group no objection to this theory, which is in fact approximately realized in several states.

(2). *These "liberalized" entrance requirements are being introduced into more and more good and large colleges and universities, and are working satisfactorily.*

I have talked with representatives of many such institutions and have found everywhere the same attitude: "We can't fight the trend! We find ways around." And so we do! I recommend that you study for example the University of Wisconsin Statutes, to learn how this is accomplished. Yet, in our faculty discussions, Wisconsin was held up to us as one of the institutions which had chosen the proper path and had seen the light.

In our University, with the tacit consent of some influential men in the administration, the department of mathematics has assumed the position that it is final authority on standards in mathematics courses offering college credit, and refuses to admit to such courses students without high school algebra and

geometry. In some Colorado institutions college credit is given even for the high school "general mathematics," and that was the original intent at our university too. The department sent a letter to all principals and superintendents in Colorado, explaining the situation. An amusing result is the comment of one high school pupil: "The situation is perfectly plain to me. I don't need to take mathematics, but I've got to have it."

(3). *Trends are justified by statistical investigations.*

Education is the only field of which I know that seems ready to accept statistics as final authority, even against experience and common sense. In a recent discussion of statistics courses in English schools in the Mathematical Gazette, the author emphasizes the eminently sound warning: "Nothing is ever *proved* by statistics, its conclusions are stated only in terms of probabilities."

Specifically emphasized in our discussions at Colorado was an experimental study which had been undertaken to determine the relationship between certain subjects taken in high school and satisfactory grades made in college. The author comes to the conclusion that "the pattern of subjects which a student took in high school has very little, if any, bearing on college success." (And on the basis of such investigations, a university official wrote to a superintendent that "there is very little evidence that high school algebra and geometry serve any useful purpose beyond satisfying entrance requirements.") This conclusion is stated to be confirmed by numerous investigations on a similar basis.

Anyone with knowledge of the situation must be dismayed at the weakness of the argument. In such experiments, students are divided into two classes, those who took a certain subject in high school, and those who did not. Unfortunately, we all know that taking a subject in high school, and mastering it, are two very different things. As I have pointed out repeatedly, a distressingly large portion of students who took and passed high school algebra cannot answer such a question as "If a yard of cloth costs a cents, what does one foot cost?"

The very arrangement of the experiment is misleading!

To draw any valid conclusion concerning the value of a high school course as preparation for college, it is obvious that students should be separated into two groups, putting into one group those who really mastered the course, and into the other those who either did not take the course or who received credit for the course without mastering it. Then, it would be important to know to what extent students were forced by their preparation to avoid certain fields and subjects in college. Neglect to take such factors into account invalidates any conclusions drawn from the tests.

On such flimsy foundations rest the arguments most frequently advanced for the "liberalization" of entrance requirements.

4. The attitude of the high school teachers. As long as I can recall, college and university teachers have taken a defeatist attitude in the fight for standards in the secondary schools. The manner in which the problem was handled at the

University of Colorado forced the mathematics department to take action. We went outside the university, and our experience in doing so forms the whole basis for my new-born optimism concerning the problem. We consulted groups of Colorado high school teachers, particularly mathematics teachers. The results were revealing:

(1) Over a hundred mathematics teachers of the Denver Section of C.E.A. protested unanimously against the changes.

(2) The Grand Junction Section, one of the other two sections in the state, sent a similar protest to the University.

(3) The Council of the Denver section sent a protest to the University.

(4) The mathematics department of Denver North High School protested unanimously.

(5) The mathematical department of Denver East High protested unanimously.

(6) The National Council of Mathematics Teachers has decided to take up the problem at its April, 1948, meeting at Indianapolis.

In addition, the following protest actions were taken:

(7) 487 of 560 students taking mathematics in the College of Arts and Science (seven of eight) objected to the changes.

(8) 13 of 15 members in one of my teacher courses, and 17 of 17 in another one, object to the change.

(9) There are indications that labor unions and P.T. organizations are beginning to realize the seriousness of the situation.

(10) The Rocky Mountain Section of the Association, in its Spring, 1947, meeting at Laramie, Wyoming, expressed itself overwhelmingly in favor of retention of algebra and geometry as entrance requirements, and as bitterly opposed to the admission of high school arithmetic and general mathematics in place of algebra and geometry. (See this MONTHLY, Vol. 54, November, 1947.)

I do not hesitate to say that the high school teachers suffer and chafe under the steady lowering of standards and are, at least in Colorado, more aware as a group of the dangers and implications of the situation, and far more willing to fight for its improvement, than are college and university faculties.

I am keenly aware of the fact that I have not even mentioned many vital problems, such as the preparation of teachers, and the depressing situation which exists in many parts of the country concerning accreditation of schools.

For the solution of all of our problems we need the whole-hearted cooperation between classroom teachers of all subjects, and on all levels, from the grades to the universities; and we shall need the advice of the colleges of education and administrative bodies, with their vast accumulated material and experience. But, those who do the actual teaching must now prepare to take over the main burden of reforms long overdue.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

THE NUMBER OF TERMS IN THE EXPANSION OF AN INFINITE DETERMINANT

A. W. GOODMAN, Rutgers University

We shall prove the following intuitively obvious result.

THEOREM. *The number of terms in the expansion of an infinite determinant has the power of the continuum.*

In other words, the group of permutations of the integers has the power of the continuum.

N. J. Lennes* settled this problem some time ago; however his proof is based on a definition of the expansion of an infinite determinant which in itself is open to comment. Let

$$(1) \qquad D = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \cdots \\ a_{21} & a_{22} \cdots a_{2n} \cdots \\ \cdot & \cdot \cdots \cdot \cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \cdots \\ \cdot & \cdot \cdots \cdot \cdots \end{vmatrix} = |a_{i,j}|, \qquad i, j = 1, 2, \cdots, n, \cdots,$$

represent both the determinant and the set of terms obtained on expansion. Let

$$(2) \qquad P = \prod_{k=1}^{\infty} a_{i_k j_k}.$$

Definition 1. P is a term of D if in the infinite product there is at most one element from each row, and at most one element from each column.

Definition 2. P is a term of D if in the infinite product there is exactly one element from each row and exactly one element from each column.

For example $a_{12}a_{24} \cdots a_{n2n} \cdots$ is a term of D by Definition 1, but is not a term of D by Definition 2. The Lennes proof is based on the first definition and provides a clever method of mapping the terms of D in a one-to-one manner on the points of an interval. We prove the theorem based on the second definition as follows.

Since we are not concerned with the convergence of the infinite product (2), we may rearrange the order of the elements according to their columns, thus

$$(3) \qquad P = \prod_{k=1}^{\infty} a_{n_k k},$$

* A Direct Proof of the Theorem that the Number of Terms in the Expansion of an Infinite Determinant is of the Same Potency as the Continuum, Bull. Amer. Math. Soc. 18 (1911) pp. 22–24.

and our definition assures us that P is an element of D if and only if the sequence $n_1, n_2, \cdots, n_k, \cdots$ is a permutation of the integers $1, 2, \cdots, k, \cdots$. We make the correspondence $P \rightarrow \xi$,

$$\xi = \sum_{m=1}^{\infty} \frac{f_m}{2^m}, \tag{4}$$

where $f_m = 1$, if $m = n_1, n_1 + n_2, n_1 + n_2 + n_3, \cdots$, and $f_m = 0$ otherwise. This provides a one-to-one mapping of the elements of D onto a subset of the interval $0 < \xi < 1$.

Secondly, we observe that every ξ, in the open interval $0 < \xi < 1$, can be written in the form (4), and uniquely so if the requirement is made that there be an infinite number of f_m different from zero. The correspondence $\xi \rightarrow P$ is made thus. For each k, the set

$$1, 2, \cdots, k, k + 1 \tag{5}$$

will have two elements $\alpha_k < \beta_k$ which are not in the set

$$n_1, n_2, \cdots, n_{k-1}. \tag{6}$$

Define

$$n_k = \begin{cases} \alpha_k, & \text{if } f_k = 1; \\ \beta_k, & \text{if } f_k = 0. \end{cases}$$

Then $\xi \rightarrow P$ is a one-to-one mapping of the interval $0 < \xi < 1$ onto a subset of the elements of D.

Now apply Bernstein's Theorem which states that if there is a one-to-one mapping of A onto a subset of B, and a one-to-one mapping of B onto a subset of A, then there exists a one-to-one mapping of A onto B. Thus D and the open interval $0 < \xi < 1$ have the same power.

This result is trivial, using Bernstein's Theorem, and the proof could have been given in many ways. The problem of constructing an effective one-to-one mapping between the set D and the open interval $0 < \xi < 1$ appears to be difficult and, as far as the writer is aware, is still unsolved.

A GEOMETRICAL FORM OF THE MULTIPLICATIVE PROPERTY OF THE DIVISOR-SUM FUNCTION

ABDUR RAHMAN NASIR, Talim-ul-Islam College, Qadian, India

1. **Introduction.** Let $\sigma(x) = \sum_{d|x} d$, so that $\sigma(x)$ denotes the sum of the positive divisors of x including 1 and x. A very simple and instructive method is investigated in the following lines to prove the multiplicative property of this function with an appeal to elementary notions of plane geometry.

We shall establish the following result.

THEOREM. *If* $(m, n) = 1$, *then* $\sigma(mn) = \sigma(m)\sigma(n)$.

Unlike the proof in classical books of the above result, the present one does not make use of the value of the function itself, which on the other hand may now be deduced from this property.

2. **Proof.** Let a_i/m, where $1 \leqq a_i \leqq m$ and $1 \leqq i \leqq p$ (say) and b_j/n, where $1 \leqq b_j \leqq n$ and $1 \leqq j \leqq q$.

Since $(m, n) = 1$, it follows that $(a_i, b_j) = 1$ for every i and j.

Hence a divisor of mn can be uniquely written in the form a_ib_j, $1 \leqq a_ib_j \leqq mn$ where i and j run through all the integers from 1 to p and 1 to q respectively.

Now take two straight lines OX and OY at right angles to each other. Cut off segments equal to divisors of m and n along OX and OY respectively, starting from O, both in the ascending order of magnitude. Clearly we can write all the divisors of mn as shown in the following scheme.

	O a_1	a_2	a_3	...	a_i	...	a_p X
b_1	a_1b_1	a_2b_1	a_3b_1	...	a_ib_1	...	a_pb_1
b_2	a_1b_2	a_2b_2	a_3b_2	...	a_ib_2	...	a_pb_2
b_3	a_1b_3	a_2b_3	a_3b_3	...	a_ib_3	...	a_pb_3
	.	.	.	.	.	.	.
	.	.	.	.	.	.	.
	.	.	.	.	.	.	.
b_j	a_1b_j	a_2b_j	a_3b_j	...	a_ib_j	...	a_pb_j
	.	.	.	.	.	.	.
	.	.	.	.	.	.	.
	.	.	.	.	.	.	.
b_q	a_1b_q	a_2b_q	a_3b_q	...	a_ib_q	...	a_pb_q
	Y						Z

Clearly all figures are rectangles except the figured designated a_1b_1 which is a square. Therefore the area of any figure is of the form a_ib_j.

In other words a divisor of mn represents the area of some figure in the above scheme.

We calculate in two different ways the area of the whole figure $OXZY$.

In the first place the area $= \sum a_ib_j = \sum_{d|mn} d$.

In the second place the area $= OX \cdot OY = \sum_{d|m} d \cdot \sum_{d|n} d$.

Hence the theorem holds.

ON A THEOREM OF GRIFFITHS

R. Goormaghtigh, Bruges, Belgium

In his note, this Monthly, vol. 54, 1947, p. 538, L. Droussent recalls that Griffiths has established that the Euler circle, circumcircle, orthocentroidal circle, and the orthoptic circle of the Steiner ellipse, of a triangle have the orthic axis as common radical axis; he further remarks, p. 540, that Griffiths' theorem

equally well could have included also the polar circle.

But Gallatly, *The modern geometry of the triangle*, p. 41, had already mentioned that the polar circle and also the circumcircle of the tangential triangle belong to the "Griffiths pencil."

So, in Droussent's theorem, the orthocentroidal circles may also be replaced by the circumcircles of the tangential triangles.

On Griffiths' theorem and the curious properties of the intersection points of the circles of the Griffiths pencil, besides the already mentioned references, see: R. Goomaghtigh, *Mathesis*, 1923, p. 475, 1933, p. 393, 1945–1946, p. 217; M. Mineur, *ibid.* 1933, p. 408, 1945–1946, p. 260; R. Deaux, *ibid.*, 1945–1946, p. 200; V. Thébault, *ibid.*, 1945–1946, p. 250.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Institute for Advanced Study

All material for this department should be sent to C. B. Allendoerfer, Institute for Advanced Study, Princeton, New Jersey.

AN ALGORITHM FOR THE DETERMINATION OF THE CONSTANTS IN THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

D. W. HALL, University of Maryland and David W. Taylor Model Basin

Suppose we are given the problem of finding the particular solution of the linear differential equation

$$G(D)x(t) = (g_0D^n + g_1D^{n-1} + \cdots + g_n)x(t) = 0, \tag{1}$$

where the coefficients are constant ($g_0 \neq 0$) and the solution is required to satisfy the conditions $x^{(i)}(b) = a_i$, ($i = 0, 1, 2, \cdots, n-1$), $x^{(i)}(t)$ denoting the ith derivative of $x(t)$.

This problem necessitates the solution of the following system of linear equations:

$$\begin{aligned} C_1e^{r_1b} + \quad C_2e^{r_2b} + \cdots + \quad C_ne^{r_nb} &= a_0 \\ r_1C_1e^{r_1b} + r_2C_2e^{r_2b} + \cdots + r_nC_ne^{r_nb} &= a_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ r_1^{n-1}C_1e^{r_1b} + r_2^{n-1}C_2e^{r_2b} + \cdots + r_n^{n-1}C_ne^{r_nb} &= a_{n-1}, \end{aligned} \tag{2}$$

where $r_1, r_2, \cdots, r_n$ are the roots (assumed to be distinct) of the characteristic equation

$$G(r) = g_0(r - r_1)(r - r_2) \cdots (r - r_n) = 0. \tag{3}$$

The system of equations (2) may be easily and quickly solved by means of the following algorithm. Neglecting the constant term of $G(r)$ we write its remaining coefficients in a row and the initial conditions in a column to the right. Columns of products are then formed and added to give the sums $f_0, f_1, \cdots, f_{n-1}$.

$$
\begin{array}{rrrrrr}
g_0 + & g_1 + & g_2 + \cdots + & g_{n-1} & & \\
\hline
a_0g_0 + & a_0g_1 + & a_0g_2 + \cdots + & a_0g_{n-1} & \vert & a_0 \\
+ a_1g_0 + & a_1g_1 + \cdots + & a_1g_{n-2} & & \vert & a_1 \\
+ a_2g_0 + \cdots + & a_2g_{n-3} & & & \vert & a_2 \\
\vdots & & & & & \\
+ a_{n-1}g_0 & & & & \vert & a_{n-1} \\
\hline
f_0 + & f_1 + & f_2 + \cdots + & f_{n-1} & &
\end{array}
$$

This defines a polynomial

$$F(r) = f_0r^{n-1} + f_1r^{n-2} + \cdots + f_{n-1}, \tag{4}$$

and the solutions of (2) are written down at once as

$$C_h = e^{-r_h t}F(r_h)/G'(r_h), \quad (h = 1, 2, \cdots, n). \tag{5}$$

Before proving (5) we give two examples to illustrate the use of the algorithm.

EXAMPLE I: Solve $(D^4 - 17D^3 + 101D^2 - 247D + 210)x(t) = 0$, where $x(0) = 1$, $x'(0) = 2$, $x''(0) = 3$, $x^{(3)}(0) = 4$.

We have at once

$$G(r) = r^4 - 17r^3 + 101r^2 - 247r + 210 = (r - 2)(r - 3)(r - 5)(r - 7),$$

$$G'(2) = (2 - 3)(2 - 5)(2 - 7) = -15, G'(3) = +8, G'(5) = -12, G'(7) = +40.$$

$$
\begin{array}{rrrrl}
1 & -17 & +101 & -247 & \\
\hline
1 & -17 & +101 & -247 & 1 \\
 & +2 & -34 & +202 & 2 \\
 & & +3 & -51 & 3 \\
 & & & +4 & 4 \\
\hline
1 & -15 & +70 & -92 & 2 \\
 & +2 & -26 & +88 & \\
\hline
1 & -13 & +44 & -4 &
\end{array}
\qquad
\begin{array}{rrrrl}
1 & -15 & +70 & -92 & 3 \\
 & +3 & -36 & +102 & \\
\hline
1 & -12 & +34 & +10 &
\end{array}
$$

$$
\begin{array}{rrrrl}
1 & -15 & +70 & -92 & 5 \\
 & +5 & -50 & +100 & \\
\hline
1 & -10 & +20 & +8 &
\end{array}
$$

$$
\begin{array}{rrrrl}
1 & -15 & +70 & -92 & 7 \\
 & +7 & -56 & +98 & \\
\hline
1 & -8 & +14 & +6 &
\end{array}
$$

$$x(t) = 4e^{2t}/15 + 5e^{3t}/4 + 3e^{7t}/20 - 2e^{5t}/3.$$

EXAMPLE II: Solve

$$G(D)x(t) = (D^n + g_1D^{n-1} + \cdots + g_n)x(t) = f(t),$$

subject to the conditions

$$x^{(i)}(0) = a_i, \qquad (i = 0, 1, 2, \cdots, n-1).$$

We first apply our algorithm to obtain a particular solution $x_1(t)$ of the homogeneous equation satisfying the initial conditions $x_1^{(0)}(0) = 0, (i = 0, 1, 2, \cdots, n-2)$ $x_1^{(n-1)}(0) = 1$. This is trivially easy from the algorithm since $F(r) = 1$. Thus

$$x_1(t) = \sum_{i=1}^{n} \frac{e^{r_i t}}{G'(r_i)}.$$

It is then a known theorem that the function

$$x_2(t) = \sum_{i=1}^{n} \int_0^t \frac{f(\sigma)e^{r_i(t-\sigma)}}{G'(r_i)}\, d\sigma$$

is a particular solution of the given differential equation, having the property that $x_2^{(0)}(0) = 0$, $(i = 0, 1, 2, \cdots, n-1)$. Thus to obtain the desired solution to our problem we need only add to $x_2(t)$ the solution $x_3(t)$ of the homogeneous equation satisfying the given initial conditions. This can be written down at once by the algorithm and the sum of $x_2(t)$ and $x_3(t)$ will be recognized immediately as that given by the Heaviside method. Appropriate modifications may of course be made if the integrand "goes bad" at the point $\sigma = 0$.

We now turn to the justification of the algorithm. It is evidently sufficient to show that it holds for C_1, by the symmetry of the problem. The function $F(r_1)$ may be written in the form

$$F(r_1) = a_0A_0^1 + a_1A_1^1 + a_2A_2^1 + \cdots + a_{n-1}A_{n-1}^1 \tag{6}$$

where

$$\begin{aligned} A_0^1 &= g_0r_1^{n-1} + g_1r_1^{n-2} + \cdots + g_{n-1} \\ A_1^1 &= g_0r_1^{n-2} + g_1r_1^{n-3} + \cdots + g_{n-2} \\ A_2^1 &= g_0r_1^{n-3} + g_1r_1^{n-4} + \cdots + g_{n-3} \\ &\vdots \\ A_{n-1}^1 &= g_0. \end{aligned} \tag{7}$$

These quantities are recognized at once as the coefficients of the polynomial $G(r)/(r - r_1)$. Thus we may write

$$G(r)/(r - r_1) = A_{n-1}^1r^{n-1} + A_{n-2}^1r^{n-2} + \cdots + A_0^1,$$

or, since r_1 is a root of $G(r)=0$,

$$G'(r_1) = A^1_{n-1}r_1^{n-1} + A^1_{n-2}r_1^{n-2} + \cdots + A^1_0. \tag{8}$$

In solving the system (2) by Cramer's rule we note that the denominator is a determinant having the value

$$H = \prod_{i<j,\,i,j=1}^{n} (r_i - r_j) = G'(r_1) \prod_{i<j,\,i,j=2}^{n} (r_i - r_j). \tag{9}$$

The numerator, when we solve for $C_1e^{r_1b}$ will be obtained by replacing the first column of H by the right members of (2). This means that the coefficient of a_j, $(j=0, 1, 2, \cdots, n-1)$ will be precisely the product of the coefficient of r_1^j in $G'(r_1)$, namely A^1_j, and the troublesome looking product

$$\prod_{i<j,\,i,j=2}^{n} (r_i - r_j).$$

This product, however, is a factor of H, by (9), and divides out when we take the quotient by Cramer's rule to obtain the exact expression given by the algorithm.

DOUBLE FACTORIALS

B. E. MESERVE, University of Illinois

The double factorial notation

$$(2n)!! = 2\cdot 4\cdot 6\cdot \cdots \cdot (2n-2)(2n) = 2^n n!$$

$$(2n+1)!! = 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)(2n+1) = \frac{(2n+1)!}{2^n n!}$$

may be considered as a generalization of $n!=1\cdot 2\cdot 3\cdot \cdots \cdot n$. It was devised to simplify the expression of the Wallis Formulas:

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2},$$

if n is even;

$$= \frac{(n-1)!!}{n!!},$$

if n is odd;

$$\int_0^{\pi/2} \sin^n x \cos^m x\,dx = \frac{(n-1)!!(m-1)!!}{(m+n)!!}\cdot\frac{\pi}{2},$$

if m, n are both even;

$$= \frac{(n-1)!!(m-1)!!}{(m+n)!!}$$

otherwise. These formulas are easily remembered and taken with the usual trigonometric substitutions are valuable time savers in many calculus problems.

Double factorials are also useful along with other multiple factorials in expressing the general terms of many series. In particular, double factorials may be used in the expression for $\Gamma(2k+1/2)$ where k is any integer and, therefore, in the Bessel functions of the first kind $J_{k+1/2}(x)$.

A REJOINDER

A. B. FARNELL, University of Colorado

A note by the author on the teaching of differential equations (this MONTHLY, vol. 54 (1947), pp. 160–162) was attacked by R. A. Johnson (this MONTHLY, vol. 54 (1947), pp. 410–411).

In my note I attempted to unify the treatment of the non-homogeneous linear differential equation with constant coefficients, using the theorem (Ince, *Ordinary Differential Equations*, Dover, pp. 73–75) that the linear differential equation

$$L\{y\} \equiv p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = r(x)$$

admits, under the assumption that $p_0(x)$, $p_1(x)$, $\cdots$, $p_n(x)$ and $r(x)$ are continuous functions of x in the interval $a \leqq x \leqq b$ and $p_0(x)$ does not vanish at any point of that interval, a unique solution which, together with its first $(n-1)$ derivatives, is continuous in (a, b) and satisfies the following initial conditions:

$$y(x_0) = y_0,\ y'(x_0) = y_0',\ \cdots,\ y^{(n-1)}(x_0) = y_0^{(n-1)},$$

where x_0 is a point of (a, b). If the coefficients depend upon a real parameter λ, and are continuous for all values of x in (a, b) where λ ranges between λ_1 and λ_2, then $y(x)$ can be proved to depend continuously on λ when λ lies within a closed interval interior to (λ_1, λ_2).

Against this the following objections were raised: first, that the general solution is put into a peculiar form, and, second, that the latter part of the theorem mentioned above has not been proved at the time linear equations are first encountered.

I restrict myself to the following justification of my suggestion: The first part of the theorem mentioned above is certainly basic in any course in differential equations whether proven or not. Further, it is necessary to use it at an early stage in such a course whether or not we like to do it. Other cases of omission of proof can usually be cited. (The same practice is of course followed in almost all mathematical subjects. It is obligatory where some idea of methods of solving problems is part of the goal, and is certainly justified at times in all courses except perhaps in advanced graduate courses.) For example, in the case where the $p_j(x)$'s are constants and we obtain by one scheme or another a solution containing n arbitrary constants, it is usually relegated to the "obvious" that we have obtained a general solution. Lastly, this existence theorem dictates a special form for the solution whether or not we choose to write it in that form.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 826. *Proposed by C. S. Ogilvy, Trinity College*

Find the equation of the ellipse with foci at $(-1, 0)$ and $(1, 0)$ and with semi-perimeter equal to the length of one arch of the sine curve, $y = \sin x$.

E 827. *Proposed by Leo Moser, University of Manitoba*

Show that the reciprocal of every integer greater than 1 is the sum of a finite number of consecutive terms of the infinite series

$$\sum_{j=1}^{\infty} 1/j(j+1).$$

E 828. *Proposed by Fritz Herzog, Michigan State College*

For positive integral x, let $P(x)$ denote the number of distinct primitive Pythagorean triangles of perimeter x. Show that $P(x)$ is unbounded; in fact, for any given non-negative integer r, the equation $P(x) = r$ has an infinitude of solutions. (Compare Problem E 812 [1948, 248].)

E 829. *Proposed by S. H. Gould, Purdue University*

Let $f(x)$ be defined in a closed interval $[a, b]$ and have the property of assuming, in any subinterval $[c, d]$, every value between $f(c)$ and $f(d)$. Prove that $f(x)$ is continuous if no rational value is infinitely often assumed.

E 830. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The six planes bisecting the adjacent dihedral angles around the base of a tetrahedron, taken four by four, form fifteen tetrahedra circumscribed about a common paraboloid of revolution.

SOLUTIONS

The Centers of Three Circles

E 539 [1942, 546]. *Proposed by Howard Eves, Oregon State College*

Give a ruler construction for finding the centers of three given linearly independent circles, no two of which are intersecting, tangent, or concentric.

(For an elucidation by the proposer, including references, see [1943, 388].)

Solution by P. H. Daus, University of California. A solution of the problem can be made to depend upon the following two ruler constructions.

CONSTRUCTION I. It is possible to construct pointwise the circle of a coaxal system through a given point.

The circles of the system determine on an arbitrary line through the point an involution. The point corresponding in the involution to the given point lies on the required circle. The involution is determined by the pairs of points in which the arbitrary line cuts the given circles and can be constructed by ruler alone. The difficulty introduced by the fact that there might be no line through the given point which cuts both circles may be avoided by first constructing pointwise a circle of the system through a point inside one circle.

CONSTRUCTION II. If one point of intersection of two circles is known, it is possible to construct pointwise a circle passing through the other (known or unknown) point of intersection.

This is accomplished by considering a variable line through the given point, and projecting its intersections with the given circles through two fixed points, one on each circle, respectively. The resulting figure for one such line is the Miquel configuration.

In terms of these constructions, the Cauer-Mierendorff solution can be reduced to the case of two intersecting circles by means of five circles constructed pointwise. A similar solution, for which there are a number of variants, can be made by means of the following four circles constructed pointwise.

Let the given non-interesting circles be c_1, c_2, c_3.

(1) Through any point A on c_1 construct the circle a coaxal with c_2 and c_3 (by I). Let the unknown intersection of a and c_1 be X. (Since the three given circles are linearly independent, a is distinct from c_1.)

(2) Through a second point B on c_1 construct the circle b coaxal with c_2 and c_3 (by I). Let the unknown intersection of b and c_1 be Y.

(3) Through any point C on b construct the circle ACX (by I).

(4) Construct the circle ACY (by II), using the pencil of lines at B. These last two circles have two known points in common.

It may be remarked here that the details of the construction (see H. Steinhaus, *Mathematical Snapshots*) when given two intersecting circles may be simplified by first applying Construction II.

An entirely different type of construction for this case than that given in the above reference is as follows. If K is a point on the radical axis of two intersecting circles c_1 and c_2, the polars of K with respect to these circles cut them, respectively, in four points on a circle k of their conjugate coaxal system. The polars of one such point with respect to c_1 and c_2 meet in a fifth point, diametrically opposite this point. The circle k is thus determined pointwise. The tangents to k (constructed by Pascal's theorem) at these four points determine the required centers.

An attempt to use these ideas directly for the case of three nonintersecting circles has not been successful.

Editorial Note. The result of this problem, coupled with the Poncelet-Steiner theorem, gives us the theorem: Any euclidean construction may be accomplished with ruler alone in the presence of three linearly independent circles; if two of the circles intersect, are tangent, or are concentric, then the third circle is redundant.

We state here, without proof, some other theorems of this character.

Any euclidean construction may be accomplished with ruler alone in the presence of (1) two circles and a point on their radical axis, (2) two circles and a center of similitude, (3) one circle and a parallelogram.

The Mathematician and the Jester

E 791 [1947, 545]. *Proposed by G. W. Walker, Buffalo, N. Y.*

The court mathematician once received his salary for a year's service all at one time, and all in silver "dollars," which he proceeded to arrange in nine unequal piles, making a magic square. The king looked, and admired, but complained that there was not a single prime number in any of the piles. "If I had but nine coins more," said the mathematician, "I could add one coin to each pile and make a magic square with every number prime." They investigated, and found that this was indeed true. The king was about to give him nine "dollars" more, when the court jester said, "Wait!". Then the jester subtracted one coin from each pile instead; and they found in this case also a magic square with every element a prime number. The jester kept the nine "dollars." How much salary must the mathematician have been receiving?

Solution by C. W. Trigg, Los Angeles City College. It is desired to form a magic square of order three from the set of primes, p_i, having the property that p_i+2 is also prime. There are 61 members of the set less than 2,000. The search is narrowed by use of the following properties:

(1) There are four conjugate pairs having the same sum.

(2) The central element, E, equals one-half each conjugate sum, *i.e.*, one-third the constant.

(3) Unless 3 or 5 appears as an element, the p_i must terminate with 1, 7, or 9.

(4) The sums of the unit's digits of the elements in each column, row, and diagonal must have the same unit's digit.

(5) Properties (3) and (4) require that all elements have the same unit's digit, or that their unit's digits form the configuration

$$\begin{matrix} 1 & 7 & 9 \\ 7 & 9 & 1 \\ 9 & 1 & 7 \end{matrix}$$

or a rotation or reflection thereof.

Within the set there are ten values of $2E$, the conjugate sum, which meet

these properties, namely: 298, 838, 1318, 1618, 1762, 2038 2098, 2122, 2182, and 2638. The first of these is the only one for which the sum of every outside column and row is $3E$. Thus the three magic squares involved are:

191	17	239	192	18	240	193	19	241
197	149	101	198	150	102	199	151	103
59	281	107	60	282	108	61	283	109

The mathematician's salary was $9E+9=9(149+1)$ or 1350 "dollars."

Remarks by Leo Moser, University of Manitoba. I would conjecture that the number of solutions is infinite, and the number of solutions with largest entry under n is $O(n/(\log n)^{18})$. This estimate is based on certain conjectural formulae of Hardy, Littlewood, and Lord Cherwell (Quarterly Journal of Math., Vol. 17, pp. 47–62, *Note on the Distribution of Intervals between PrimeNumbers*). I would also conjecture that there exist magic squares of this type of any order not less than three. An example of such a square of order four is

29	1061	179	227
269	137	1019	71
1049	101	239	107
149	197	59	1091

Of course, proving these conjectures would definitely not be an "elementary" problem, as they would imply the existence of an infinity of prime pairs, proof of which is one of the outstanding unsolved problems of number theory.

Also solved by Josephine and Richard Andree, A. S. Anema, W. E. Buker, Joseph Clare, Monte Dernham, R. L. Greene, Frank Herlihy, Roger Lessard, Leo Moser, Bart Park, S. T. Parker, and the proposer.

Buker called attention to the discussion of magic squares, whose elements are primes, on p. 125 of Dudney's *Amusements in Mathematics.*

Lessard claimed that the next solution to the given problem is

6198	5502	3030
2142	5010	7878
6690	4518	3822

In this case the mathematician's salary would have been 45090 "dollars"!

Shuffling Cards

E 792 [1947, 545]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Show that a pack of $2n$ cards is brought back to its original position in at most $2n-2$ perfect riffle shuffles. (A perfect riffle shuffle is one which sends the cards from the arrangement $1, 2, 3, 4, \cdots, 2n-1, 2n$ to the arrangement $1, n+1, 2, n+2, \cdots, n, 2n$.)

Solution by the Proposer. Let $m=2n-1$. It is obvious that a perfect riffle shuffle sends the card in position x to position y where $y\equiv 2x-1 \pmod{m}$. After k shuffles the card in position originally x moves to position $2^kx-(2^k-1) \pmod{m}$. If after k shuffles each card is in its original position, then $2^kx-(2^k-1) \equiv x \pmod{m}$ for all x. Hence $(2^k-1)(x-1)\equiv 0 \pmod{m}$ for all x or $2^k-1\equiv 0 \pmod{m}$. Hence the least number of shuffles needed to bring the pack to its original position is the minimum $k>0$ for which $2^k-1\equiv 0 \pmod{m}$. Since m is odd, $(2\ m)=1$, whence $2^{\phi(m)}-1\equiv 0 \pmod{m}$. Therefore, the minimum $k<\phi(m) \leq m-1=2n-2$. Note that for a pack of 52 cards we have $k=8$, a remarkably small number.

Also solved by J. B. Kelly, Roger Lessard, and Leo Moser. Moser pointed out that this problem is essentially the same as that solved on pp. 244–245 of *Elementary Theory of Numbers* by Uspensky and Heaslet. The riffle shuffle on $2n$ cards leaves the top and bottom cards fixed and shuffles the remaining cards in the manner there described. The number of shuffles required to regain the original is therefore the exponent to which 2 belongs mod $(2n-1)$. This is a divisor of $\phi(2n-1)$ and hence not greater than $2n-2$.

Irving Kaplansky stated that the problem is treated in *Math. Gazette*, vol. 15 (1930), pp. 17–20.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4305. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$1+\left(\frac{1+\frac{1}{2}}{2}\right)^2+\left(\frac{1+\frac{1}{2}+\frac{1}{3}}{3}\right)^2+\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{4}\right)^2+\cdots=\frac{17\pi^4}{360}.$$

4306. *Proposed by Paul Erdös, Syracuse University*

Let there be given eight arbitrary points in space. Prove that one can always select three of them which do not form an acute-angled triangle.

4307. *Proposed by Emma Lehmer, Berkeley, California*

Let p be a prime number of the form $p=3\cdot 2^a k+1$, and let n be any number not divisible by 3 or 2^a. Show that there are numbers a, b, independent of n, such that a, b, $a+1$, $b+1$ are nth power residues modulo p.

4308. *Proposed by H. S. Wall, University of Texas*

Let $f(z)$ be a function of the complex variable z, such that

$$f(-w^2) = -\frac{F(w)}{w},$$

where $F(w)$ has the properties: (a) $F(w)$ is an odd function of w, (b) $F(w)$ is analytic and $I[F(w)]\leqq 0$ for $I(w)>0$. Show that $I[f(z)]\leqq 0$ for $I(z)>0$.

4309. *Proposed by R. Goormaghtigh, Bruges, Belgium*

Let ABC be a triangle, D, E, F the contact points of one of the tritangent circles with BC, CA, AB, respectively, and let A', B', C' and D', E', F' be the projections of a point M of that circle on BC, CA, AB and EF, FD, DE. Show that the lines joining the projections of M on $A'E'$ and $A'F'$, $B'F'$ and $B'D'$, $C'D'$ and $C'E'$ are concurrent.

SOLUTIONS

Five Parallel Forces in Equilibrium

4080 [1943, 264]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given five points in space, if five parallel forces applied at these points are in equilibrium, then equilibrium results also when each force is translated to the center of the circumsphere of the tetrahedron which has for vertices the points of application of the four other forces.

Note. The similar problem for four points in a plane was proposed by J. Neuberg, *Educational Times, Reprints*, 1891, v. 55, p. 82.

*Solution by R. Blanchard, Le Havre, France.** Let us demonstrate first the following theorem: *In a tetrahedron the barycentric coördinates of a point with respect to the pedal tetrahedron of this point are the same as the barycentric coördinates of its isogonal conjugate with respect to the given tetrahedron.*

Consider a tetrahedron $ABCD$, two points P_1 and P_2 isogonally conjugate with respect to it, and $A_1B_1C_1D_1$ the pedal tetrahedron of P_1, and let x, y, z, t be the normal coördinates of P_1 with respect to $ABCD$. The barycentric coördinates of P_1 with respect to $A_1B_1C_1D_1$ are proportional to $yzt\ \sigma_a, \cdots$, where $\sigma_a, \cdots$, designate the sines [1] of trihedral angles supplementary to the trihedral angles of $ABCD$. The normal coördinates of P_2 with respect to $ABCD$ are proportional to $yzt, \cdots$, and its barycentric coördinates are proportional to $S_a yzt, \cdots$, where $S_a, \cdots$, designate the areas of the faces of $ABCD$. The

* Translated and checked by W. E. Byrne, Virginia Military Institute.

theorem follows from the proportionality of $\sigma_a, \cdots,$ and $S_a, \cdots,$ (Dostor's relation).

Now let us designate by A, B, C, D, E the five given points and by $0, 0_a, 0_b, 0_c, 0_d$ the circumcenters of the tetrahedra $ABCD, EBCD, ECDA, EDAB, EABC$. The problem is equivalent to that of proving that E has the same barycentric coördinates with respect to $ABCD$, that O has with respect to $O_aO_bO_cO_d$ [2]. Let $A'B'C'D'$ be the antipedal tetrahedron [3] of E with respect to $ABCD$ and E' the isogonal conjugate of E with respect to $A'B'C'D'$ [4]. On the one hand E' has the same barycentric coördinates with respect to $A'B'C'D'$ that E has with respect to $ABCD$. On the other hand, O_a, O_b, O_c, O_d, O are the transforms of A', B', C', D', E' under the homothety $(E, \frac{1}{2})$. We conclude that O has the same barycentric coördinates with respect to $O_aO_bO_cO_d$ as E has with respect to $ABCD$.

J. Neuberg gave the same generalization of his proposition concerning four coplanar points. See *Educational Times, Reprints*, v. 54, 1891, p. 30.

Notes by the Translator: 1. The sine of a trihedral angle is the triple scalar product of the three unit vectors laid off from the vertex along the three edges.

2. For an explanation of the theorem of statistics attached to the barycentric coördinates of Möbius, see Darboux, *Principes de Géométrie Analytique*, Ch. II.

3. The antipedal tetrahedron of E with respect to $ABCD$ is the tetrahedron formed by planes drawn through A, B, C, D perpendicular to EA, EB, EC, ED, respectively.

4. If two points are isogonally conjugate with respect to a tetrahedron, they have the same pedal sphere, whose center bisects the line segment joining them. Hence O is the midpoint of EE'. See Coolidge, *A Treatise on the Circle and the Sphere*, p. 233.

An Inequality Connected with n!

4226 [1946, 594]. *Proposed by Paul Erdös, Syracuse University*

Let $n! = \prod_{p \leq n} p^{k_p}$ (p, primes). Assume that $p_1 > p_2$ and $k_{p_1} < k_{p_2}$. Prove that $p_1^{k_{p_1}} < p_2^{k_{p_2}}$.

Solution by W. J. Harrington, Pennsylvania State College. The exponent of the highest power of a prime p contained in $n!$ is given by

$$\nu(n) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots, \tag{1}$$

where $[Q]$ denotes the greatest integer $\leqq Q$.* The following well known properties of $\nu(n)$ will facilitate the solution

$$\nu(n) < n \sum_{j=1}^{\infty} 1/p^j = n/(p-1). \tag{2}$$

$$\nu(n_1 + n_2) \geqq \nu(n_1) + \nu(n_2). \tag{3}$$

* Uspensky and Heaslet, Elementary Number Theory, pp. 99–100.

For positive integers m and c,

$$\nu(mp) = m + \nu(m), \tag{4}$$

and

$$\nu(cn) \geq c\nu(n). \tag{5}$$

If m is defined by

$$mp \leq n < (m+1)p,$$

m being a positive integer, it follows from (4) that

$$\nu(n) \geq \nu(mp) = m + \nu(m).$$

If $m \geq p$, then $\nu(m) \geq 1$ and $\nu(n) \geq m+1$. If $1 \leq m \leq p-1$, then $\nu(n) \geq m$. Since in each case $n \leq (m+1)p-1$, we have

$$\nu(n)/n > 1/p, \qquad n \geq p^2, \tag{6}$$

$$\nu(n)/n \geq m/[(m+1)p-1] \geq 1/(2p-1), \qquad n < p^2. \tag{7}$$

To handle the proposed problem, let

$$p_1 = cp_2 + b, \tag{8}$$

where $c \geq 1$ and $1 \leq b \leq p_2 - 1$. We shall write k_i for k_{p_i} and $\nu_i(n)$ for $\nu(n)$ relative to the prime p_i. From (2) we have

$$n > k_1(p_1 - 1), \tag{9}$$

and thus from (3), (4) and (8) we have

$$k_2 = \nu_2(n) \geq \nu_2(k_1cp_2 + k_1b - k_1) \geq k_1c + \nu_2(k_1c) + \nu_2(k_1b - k_1). \tag{10}$$

Consider separately the two cases $c \geq 2$ and $c = 1$.

Case I. $c \geq 2$.

From (10) we have $k_2 > k_1c$, and thus $p_2^{k_2} \geq (p_2^c)^{k_1}$, whence our desired result is established if $p_2^c > p_1$. From (8), $p_1 \leq (c+1)p_2 - 1$, and hence we consider the inequality.

$$p_2^c > (c+1)p_2 - 1. \tag{11}$$

For all $p_2 \geq 3$ with $c \geq 2$ and for all $p_2 \geq 2$ with $c \geq 3$, (11) holds and the problem is solved. The special case $p_2 = c_2 = 2$ requires $p_1 = 5$ by (8). From (9) and (10),

$$k_2 \geq 2k_1 + \nu_2(2k_1) \geq 2k_1 + k_1 = 3k_1$$

and thus

$$p_2^{k_2} \geq (p_2^3)^{k_1} = 8^{k_1} > p_1^{k_1}.$$

Case II. $c = 1$.

We notice that the problem is equivalent to that of showing

$$\log p_1 / \log p_2 < k_2/k_1. \tag{12}$$

From (8) with $c=1$, and (12) it follows that it will be sufficient to show

$$k_2/k_1 \geqq 1 + b/p_2 \log p_2, \tag{13}$$

which, with the aid of (10) and (5), becomes

$$k_2/k_1 \geqq 1 + b\nu_2(k_1)/k_1. \tag{14}$$

The investigation is continued by considering separately $k_1 \geqq p_2^2$, $p_2 \leqq k_1 < p_2^2$, and $k_1 < p_2$.

Case II(a). $c=1$, $k_1 \geqq p_2^2$.

From (6) and (14), (13) can be established for $p_2 \geqq 3$. For the remaining case, $p_2=2$, write $k_1 = 2h_1 + h_0$ where $h_1 \geqq 2$ and $h_0 = 0$ or 1. The proof of (12) for the case $h_1 = 2$, $h_1 = 3$, $h_1 \geqq 4$ presents no difficulty.

Case II(b). $c=1$, $p_2 \leqq k_1 < p_2^2$.

Take $mp_2 \leqq k_1 < (m+1)p_2$ with $1 \leqq m \leqq p_2 - 1$. From (7) and (14),

$$k_2/k_1 \geqq 1 + mb/[(m+1)p_2 - 1] \geqq 1 + b/(2p_2 - 1).$$

This establishes (13) for $p_2 \geqq 7$ and for $p_2 \geqq 5$ with $m \geqq 2$. Special cases remaining are

$$p_2 = 5, \quad m = 1; \qquad p_2 = 3, \quad m = 1 \text{ or } 2; \qquad p_2 = 2, \quad m = 1,$$

for each of which there is no difficulty in establishing (12).

Case II(c). $c=1$, $k_1 < p_2$.

Here $n \geqq k_1 p_1 = k_1 p_2 + k_1 b$, and hence

$$\nu_2(n) \geqq k_1 + \nu_2(k_1 b).$$

For $k_1 b \geqq p_2$, we have by (7)

$$k_2/k_1 \geqq 1 + b\nu_2(k_1 b)/k_1 b \geqq 1 + b/(2p_2 - 1)$$

and (13) is established for $p_2 \geqq 7$. As in case II(b), (12) can readily be established for the special cases $p_2 = 5$, 3, and 2. Also for $k_1 b < p_2$ there is no difficulty in establishing (12).

Developables of a Congruence

4238 [1947, 112] *Proposed by C. E. Springer, University of Oklahoma*

At each point of the revolute

$$x = u \cos v, \qquad y = u \sin v, \qquad z = \phi(u)$$

a straight line generator of a congruence is taken with direction cosines proportional to

$$\cos v, \qquad \sin v, \qquad \phi' - u\phi'',$$

where primes indicate differentiation with respect to u. Show that the developables of the congruence intersect the revolute in its lines of curvature.

Solution by V. G. Grove, Michigan State College. The problem as stated is a special case of a more general one. Let the generators of the congruence have direction cosines proportional to X, Y, Z defined by expressions

$$X = \psi(u, v) \cos v, \qquad Y = \psi(u, v) \sin v, \qquad Z = \theta(u, v).$$

The pairs of functions (X, x), (Y, y), (Z, z) may be shown by elementary methods to be solutions of the following system of differential equations

$$X_u = M_1^1 x_u + M_1^2 x_v + E_1 X,$$
$$X_v = M_2^1 x_u + M_2^2 x_v + E_2 X,$$

wherein M_β^α are defined as follows and E_1, E_2 are not material for our purposes:

$$\Delta M_1^1 = u(\theta\psi_u - \theta_u\psi), \qquad \Delta M_1^2 = 0$$
$$\Delta M_2^1 = u(\theta\psi_v - \psi\theta_v), \qquad \Delta M_2^2 = +\psi(\theta - \psi\phi')$$
$$\Delta = u(\theta - \phi'\psi).$$

The meridians (v=const) and parallels (u=const) of the revolute are known to be the lines of curvature. Moreover the developables of the congruence intersect the revolute in the parametric curves if and only if $M_2^1 = M_1^2 = 0$. (See V. G. Grove, *A general theory of surfaces and conjugate nets.* Trans. Am. Math. Soc. V. 57 (1945) p. 106.) Hence the congruence we are discussing has the desired property if and only if $M_2^1 = 0$, $\Delta \neq 0$, that is if and only if $\theta = \psi U(u)$. The direction numbers of the generators of the congruence may be written in the form

$$\cos v, \qquad \sin v, \qquad U(u), \qquad U \not\equiv \text{const}, \qquad U \not\equiv \phi',$$

$U(u)$ being otherwise an arbitrary function of u. The function $U = \phi' - u\phi''$ in the problem is a special case.

Also solved by P. D. Thomas, and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Elements of Mathematical Statistics. By C. V. L. Charlier. Translated and published by J. A. Greenwood (25 Winthrop Street, Brooklyn 25, New York), 1947, 120 pages. $3.00.

This translation follows essentially the 1920 edition. Part One, which consists of eight chapters, deals with homograde statistics; that is, with data in which each individual who has been measured with respect to an attribute either possesses (or is assumed to possess) the attribute to the same degree of intensity. Successive chapters treat the arithmetic mean, the dispersion, the mean error, the theorems of Bernoulli, Poisson, and Lexis, and observed statistical data.

Part Two, which consists of seven chapters, is concerned with heterograde statistics; that is, data in which the degree of intensity of a measured attribute is of concern. There are chapters on the Normal, the Type A, and the Type B frequency curves; on correlation; and on certain abridged methods of calculation.

There is an appendix which consists of four sets of tables, a bibliography, and an index. The reader should consult the translator's preface concerning both the tables in the appendix and those in the text.

This reviewer believes the translator should be commended not only for making this monograph more available to beginning students, but also for his work on the tables.

A. T. CRAIG.

College Algebra. By E. R. Heineman. New York, The Macmillan Company, 1947. 9+359 pages. $3.25.

The reviewer holds the opinion that, in undergraduate courses in mathematics, very little, if any, classroom time should be spent in discussion of new material, the student being required to "dig it out" for himself. Especially in elementary courses, such a plan requires the use of a text which is clear in its presentation, and which contains good illustrative examples and apt warnings about common errors. The text under consideration is especially good in this respect.

It is of the standard type of college algebra text, devoting 22 chapters to the topics usually classified as college algebra. Texts covering a variety of topics are often well-done in parts and quite the opposite in other portions. The reviewer was interested to note how consistently he found this text to treat topics in the fashion his experience had taught him to prefer. In only two instances was there disappointment in this regard: (1) In presenting the factoring method of solving equations the following principle *alone* is stated: "If the product of two or more quantities is zero, then at least one of the factors must be zero." In examining sixteen recent texts, the reviewer found three which also confuse the issue by not stating this principle in reverse. (2) In treating multiplication of fractions, illustrations of products are given in which a factor of the numerator of one fraction is cancelled against the factor in the denominator of another, no justification for this being given. Although the text is superior in its treatment of incorrect cancelling in fraction simplification, the student having trouble in this matter will not be helped by these illustrations.

The presentation of conditional equations and identities is the best the reviewer has seen. The treatment of mathematical induction is also outstanding. There are instances throughout the book of helpful cautionings and suggestions which are not often found in texts. Examples of these: The student is warned not to rule out a number as a possible rational root of a depressed equation just because it was found to be a root of the original equation. There is an illustration of the importance of indicating clearly the main division line in certain complex fractions.

Few errors of a typographical nature were noted, the only one of any consequence being the omission of the words "of the logarithm" in the theorem giving the characteristic of a logarithm.

This book will rank along with the author's *Plane Trigonometry* (McGraw-Hill, 1942) as a successful text.

K. W. WEGNER

NEW BOOKS RECEIVED

Mathematical Table Makers (Scripta Mathematica Studies, No. 3). By R. C. Archibald. New York, Scripta Mathematica, 1948. 82 pages. $2.00.

Modern-School Geometry. New edition. By R. Schorling, J. R. Clark, and R. R. Smith. Yonkers, New York, World Book Co., 1948. 12+436 pages. $1.88.

Tables of the Bessel Functions of the First Kind of Orders Sixteen through Twenty-seven. Prepared by the Staff of the Computation Laboratory of Harvard University. Cambridge, Harvard University Press, 1948. 764 pages. $10.00.

The Works of the Mind. By R. B. Heywood. Chicago, University of Chicago Press, 1947. 11+246 pages. $4.00.

Methods of Algebraic Geometry. Vol. I. By W. V. D. Hodge and D. Pedoe. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 8+440 pages. $6.50.

Basic Mathematics: A Workbook. By M. W. Keller and J. H. Zant. Boston, Houghton-Mifflin Co., 1948. 4+253 pages. $1.50.

Algebre et Analyse Lineaires. By A. Lichnerowicz. Paris, Masson, 1947. 316 pages. 800 fr.

Analytic Geometry. Fourth Edition. By C. E. Love. New York, The Macmillan Co., 1948. 11+306 pages. $3.50.

Fundamentals of Statistics. By J. B. Scarborough and R. W. Wagner. Boston, Ginn and Co., 1948. 7+145 pages. $2.40.

Projective and Analytical Geometry. By J. A. Todd. New York, Pitman Publishing Co., 1948. 10+289 pages. $4.50.

Analytic Geometry. Revised Edition. By Roscoe Woods. New York, The Macmillan Co., 1948. 14+322 pages. $3.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Summary of Papers Reported by Clubs

A study was made of the papers presented at mathematics club meetings, as reported in this MONTHLY, for the period from January, 1943, to December, 1947, inclusive. A total of 652 papers was reported for that period with 183 somewhat different titles. It was deemed inadvisable to record here all of the different titles reported, so that, for summary purposes, many were grouped together under related headings. The number of papers reported on some of the most popular topics include:

Men and women in mathematics	55
Mathematical recreations	54
Astronomy, meteorology, navigation and applications	46
History and teaching of mathematics	36
Mathematical systems	35
Application of mathematics to war	35
Calculating machines and other aids	30
Topics related to probability and finite differences	29
Various geometries	23
Geometric figures and constructions	23
Discussion of various curves	22
Statistical and financial applications	21
Greek contributions to mathematics	21
Topological aspects	20
Modern algebra	17
Theory of numbers	16
Cryptography	15
Mathematics in arts and sciences	15
Foundations and logical concepts	14
Fourier and other series	12

Club Reports, 1946–47

Pi Mu Epsilon, University of Arkansas

The *Pi Mu Epsilon* Chapter of the University of Arkansas reports that initiation ceremonies were held for both fall and spring semesters of 1946–47. Two formal programs were given; Melvin Lieberstein talked on *Mathematics and logical systems* and D. P. Richardson talked on *The history of the calendar*.

Officers for the year 1946–47 were: President, James L. Fischer; Secretary, Lillia Dewees; Treasurer, Hartman Hotz.

Officers elected for the year 1947–48 are: President, M. A. Lilly; Vice-President, Maclyn McKeehan; Secretary, H. P. Hotz; Treasurer, Webb Henderson.

Professor D. P. Richardson was elected permanent secretary.

Mathematics Club, Connecticut College

The *Mathematics Club* of Connecticut College held four meetings during the school year 1946–47, some of which were recreational as well as educational.

The officers for 1946–47 were: President, Shirley Bodie; Secretary-Treasurer, Edith Techner; Committee Chairmen, Roberta Richards, Virginia Doyle, and Jacqueline Greenblatt.

Officers elected for 1947–48 are: President, Roberta Richards; Secretary-Treasurer, Virginia Doyle; Committee Chairmen, Eleanor Penfield, Jean Webber, and Janet Johnson.

Kappa Mu Epsilon, Kansas (Pittsburg) State Teachers College

The *Kansas Alpha* Chapter of *Kappa Mu Epsilon* conducts open meetings once a month, to which anyone interested is invited. Two of these meetings were picnics, one in October and the other in May. Ten new members were initiated in July, 1946, and twenty-three in February, 1947. This makes a total of 541 members initiated into *Kansas Alpha* Chapter. Topics presented by members were:

Lost and rediscovered mathematics
Unsolved mathematics
Addition-subtraction logarithms
Mathematics in the Navy
Curves used in civil engineering
Ancient methods of calculating the value of π
Modern methods of calculating π
Scientific and mathematical satire in Gulliver's Travels
The number 2 in mathematics
Report of K. M. E. National Convention.

The chapter awards Kappa Mu Epsilon keys annually to the two seniors making the highest grade point average in their mathematics courses. James Standley and Donna Stewart of the class of 1947 were the recipients of these awards.

Officers for 1946–47 are: President, James Standley; Vice-President, Jack Lambert; Secretary, Billie Schultz; Treasurer, Donna Stewart; Corresponding Secretary, Professor J. A. G. Shirk. Professor R. G. Smith, as the new head of the Department of Mathematics, becomes sponsor of the chapter. He succeeds Professor Shirk, who retired as Head of the Department after serving thirty-two years.

Pi Mu Epsilon, Oklahoma A. & M. College

The *Oklahoma Beta* Chapter of *Pi Mu Epsilon* made a very successful attempt in restoring the mathematical interests of students to prewar level.

The annual spring banquet and initiation was held on May 15, at which time fifty-two students and faculty members were initiated. The group was addressed by K. S. Chester, Head of the Botany and Pathology Department and

Director of the College's Research Foundation. His topic was *Measurements, estimates, and guesses.*

The annual *Pi Mu Epsilon* scholarship award to the outstanding freshman in Mathematics was presented at the Honors Day Convocation to Larry Gene Sigler.

During the year the following lectures were given:

Fixed point theorems, by Dr. O. H. Hamilton

Integral distances, by Robert R. Reynolds

Magic squares, by Raymond L. Caskey.

The officers for 1945–47 were: President, Roscoe Jones; Vice-President, Margaret Ann Reiff; Faculty Adviser, Professor J. C. C. McKinsey.

The officers elected for 1947–48 are: President, Robert D. Morrison; Vice-President, Philip J. Miller; Secretary-Treasurer, Helen Dayton; Faculty Adviser, Professor J. C. C. McKinsey.

Graduate Mathematics Club, Indiana University

The *Graduate Mathematics Club* was organized at Indiana University in the fall of 1946. Members of the Mathematics Department gave talks on various fields of mathematics. The topics were:

Mathematical quotations from non-mathematical sources, by M. A. Zorn

Antipodal theorems, by S. Eilenberg

The problem of Kakeya, by J. W. T. Youngs

The billiard ball problem, by D. Gilbarg

The Bernoullian numbers, by H. S. Vandiver

The motion of the perihelion of Mercury, by K. P. Williams.

A picnic was held in a nearby State Park at the beginning of June.

The Executive Committee for the year was composed of Mrs. Lee Pruett, Gordon Overholtzer, and I. M. Herstein.

Delta-X, University of the City of Toledo

In the past year, *Delta*-X has maintained a membership of sixty active members. The annual get-acquainted roast opened the social activities of the club early in September. A Christmas party was combined with the regular December meeting and a pot luck supper was held before the March business meeting. The annual *Delta*-X banquet was held in May at which Professor Edward Ebert spoke on *Inequalities*. *Delta*-X yearbooks were presented to members at the banquet. The last function of the club was the annual spring picnic which was held in June.

The following papers of interest to students of mathematics were presented at regular monthly meetings:

The geometry of paper folding, by Jane McFillen

The development of the calendar, by Wayne McQuillin

The geometry of soap bubbles, by George Messomore

The theory of probability, by Sam Part

Nomograms, by Richard Reisbach.

Officers for the past year were: President, Lois Zeigler and Mary Novick; Vice-President, Sam Part; Secretary-Treasurer, Shirley Remmert.

Newly elected officers are: President, Fred Miller; Vice-President, Wayne McQuillin; Secretary-Treasurer, Martha Goodwin. Professor Wayne Dancer is adviser of the club.

Pi Mu Epsilon, University of California

At the fall initiation banquet in December 1946, sixteen new members were initiated; all sixteen were students and teaching assistants of the mathematics department.

The spring initiation, held in April, 1947, consisted of a picnic at the John Hinkle Park and evening entertainment in the clubhouse.

During the year, the following mathematical papers were presented at the various *Pi Mu Epsilon* meetings:

Extension of the definition of exponents to irrational numbers, by Bjarni Jonnson

The large computing units of the war, by D. H. Lehmer

Trilinear coordinates of modern geometry, by F. Harary

Survey of elementary potential theory, by Fred Newstadter

Symbolic logic in real variables, by F. Wolf

Cantor's definition of the real numbers in terms of fundamental sequences of rationals, by Alan Davis

Suspension bridge flutter, by Edmund Pinney

Identities in the theory of partitions, by Henry Alder.

The officers for the academic year, 1946–47, were: Director, Henry Alder; Vice-Director, Mrs. Alan Davis; Secretary, Mrs. Jack Gysbers; Treasurer, Hewitt Kenyon; Librarian, Miss Sarah Hallam.

The officers for the year 1947–48 are: Director, Miss Barbara Vernon; Vice-Director, Hewitt Kenyon; Secretary, Miss Edith Wetzel; Treasurer, Ralph Willoughby; Librarian, Miss Sarah Hallam.

Pi Mu Epsilon, Hunter College

The topics considered by the *New York Beta* Chapter of *Pi Mu Epsilon* during the year, 1946–47, were: the theory of higher plane curves, and certain functions of a complex variable and the mapping of the complex plane determined by these functions.

The students who presented papers were Sally Rothstein, Florence Salom, Helvie Thors Impola, Clara Rogovin, Jean Ruderman, Ina Pinchuck, Cecile Salwen, Joyce Marrits, Fay Kleiner, Annette Drucker, Blanche Marten, Mildred Vogt, Anna Ferrara, Florence Myres, Wilhelmina Fluhr, Esther Silver Chad, Roslyn Tarnell, Dorothy Beck, and Marianne Weisz.

Nineteen new members were elected to the Chapter during the year.

At the fall initiation the speaker was Professor Lester Hill of the Hunter College Department of Mathematics who discussed *Teaching mathematics for the United States Army in the Biarritz experiment.*

The guest speaker at the spring initiation was Professor E. R. Lorch of Columbia University whose topic was *The foundations of Euclidian geometry.*

Chapter members participated in the Second Annual Intercollegiate Mathematics convention sponsored by the Brooklyn College, Hunter College, and New York University Chapters of the fraternity. This affair was held on April 19 in Boylan Hall of Brooklyn College. The guest speakers of the evening were P. A. Smith of Columbia University who discussed *Topics in topology* and E. C. Molina of the Newark College of Engineering who spoke on *Applications of probability.*

Prizes for the best papers presented to the society were given to Mary Greene, Cecile Salwen, and Mildred Vogt.

The officers for the year, 1946–47, were: President, Cecile Salwen; Corresponding Secretary, Ruth Friedman; Recording Secretary, Annette Drucker; Treasurer, Mildred Vogt; Director, R. Lucile Anderson.

Newly elected officers are: President, Efrosene Josephides; Corresponding Secretary, Alexa Drexler; Recording Secretary, Marian Boykan; Treasurer, Anna Lemont; Director, Marguerite Darkow.

Mathematics Club, Kansas State College

In addition to two social events, the members of the *Mathematics Club* of Kansas State College heard the following papers discussed:

Methods of graphing, by James Smith

Nomographs, by Robert Kromhout

The operational aspect as the foundation of mathematics, by Elizabeth Button

Topology, by Thirza Mossman and Robert Reinking

Vector methods as applied to trigonometry, by F. B. Sloat

Probability, by Prof. H. C. Fryer.

Officers for the year 1946–47 were: President, Joe Ludlum, Jr.; Vice-President, C. S. Clay; Secretary, Elizabeth Button; Treasurer, Robert Reinking; Program Chairman, Marn Johnson; Sponsor, Prof. C. F. Lewis.

Zeno Club, Alfred University

Among the most interesting papers presented to the *Zeno Club* of Alfred University for 1946–47 were:

Probability in Gin Rummy, a very successful and original paper by John Freund

Some special methods of solving equations, by Dr. A. E. Whitford

Fermat's last theorem, by Marion Miller

Mathematical series, by Frank Olson.

The 1946–47 officers were: President, Marion Miller; Vice-President, Juli-

anne Sanford; Secretary-Treasurer, Joan Berkman.

Those elected for 1947–48 are: President, Ralph Jordan; Vice-President, Joan Berkman; Secretary-Treasurer, Mary van Norman.

Mathematics Club, Butler University

Two social events, a Christmas party and the Annual Spring picnic, were held by the *Mathematics Club* of Butler University during 1946–47. Papers presented include:

Mathematical fallacies, by Marianne Buschmann and Audrey Klein

What is mathematics? by Reba Marshall, leader of round-table discussion.

Mathematical instruments, by Edward O'Nan, Mrs. Mary Bilby, and William Farmer

Short cuts in computing, by Mr. W. R. Krickenberger of Technical High School

Mathematics in meteorology, by Mr. Williamson of U. S. Weather Bureau

Theory of relativity, by Mrs. Juna L. Beal

The art of map making, by Mr. Walter Gingery of Washington High School

Anti-aircraft, by Miss Elna J. Hilliard

Atomic fission, by Mr. Donald G. Wittig

Men of mathematics, by Clifford Wagoner and William Fuller.

Officers for 1946–47 were: President, Betty Beck; Vice-President, Audrey Klein; Secretary, Elsie Popplewell; Treasurer, Reba Marshall; Sponsor, Mrs, Juna L. Beal.

Officers for 1947–48 are: President, Dorothy Reinacker; Vice-President, William Fuller; Secretary, Ethel Jackson; Treasurer, Clifford Wagoner.

Mathematics Club, Ball State Teachers College

The material presented to the *Mathematics Club* of Ball State Teachers College during 1946–47 dealt with the role of mathematics in modern warfare as observed by the veterans presenting the papers:

The general problems encountered in preparation for and during a flight, by Roy Tussey

Weather observing, by Ernest McClain

The assembly of data and the drawing of maps, by Clay Babcock

Problems encountered as chief forecaster on a Pacific island base, by Ed Shreve

Use of the maneuvering board in fleet operations, by Fred Deal

Experiences at the Bikini atom bomb tests, by Clarence Oswald

Mathematics in field artillery, by Clarence Buesking and Kenneth Paucher

Radar, by William Miner, a former club member.

Officers elected for 1947–48 are: President, Joe Scherrer; Vice-president, Kenneth Paucher; Secretary, Ed Shreve; Treasurer, Iraida Reed; Program Chairman, Joe Jackson.

Mathematics Club, New York State College for Teachers

The meetings of the club were held monthly, featuring one or two speakers each meeting. The topics discussed were:

Bridges and unicursal curves, by E. A. Butler, Instructor of Mathematics

The curve of the devil, by Herbert Weiner

Puzzles, by Louise Dodge

Clerical and mechanical aspects of teaching, by C. J. Haughey, Instructor and Supervisor of Mathematics

Derivation of trigonometric formulas by geometry, by Marilyn Burnup

History of verbal problems, by Doris Quinn

Magic squares, by Marie Balfoort

The oddities of numbers, by Eleanor Merritt.

The officers for the year 1946–47 were: President, Ruth Seelbach; Vice-president, Betty Whitney; Secretary, Florence Mace; Treasurer, Elsa Moberg. The officers for the year 1947–48 are: President, Eleanor Merritt; Vice-president, Betty Whitney; Secretary, Elsa Moberg; Treasurer, Ruth Marschner.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

RESOLUTIONS OF THE NATIONAL COUNCIL

At the Annual Meeting of the National Council of Teachers of Mathematics on April 2–3, 1948 the following resolution was adopted: The membership of the N.C.T.M. commends the Mathematical Association of America in its recent adoption of resolutions stating their desire for close cooperation with the N.C.T.M. in the efforts of both organizations to improve the coordination of high school and college mathematics.

THE PHILOSOPHY OF SCIENCE ASSOCIATION

The Philosophy of Science Association has been reorganized with Philipp Frank of Harvard University as president; C. W. Churchman of Wayne University is secretary-treasurer. Applications for membership may be sent to Professor Churchman. Dues are $5.00 per year.

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

An International Congress of Mathematicians will be held in Cambridge, Massachusetts, in 1950 under the auspices of the American Mathematical

Society. The Society originally planned to act as host for a Congress in September, 1940, which was also scheduled to meet in Cambridge. At the 1936 Congress in Oslo, Norway, the invitation for the 1940 Congress was issued by the American delegation in the name of the American Mathematical Society. Plans for the 1940 Congress were practically completed when the outbreak of World War II in September, 1939, made it necessary for the Society to postpone the Congress to a more favorable date. An Emergency Committee was established to carry on in the interim and, on recommendation of this Committee, the Council of the Society voted to hold the Congress in 1950.

The 1950 Congress will be the third International Congress of Mathematicians to be held on the continent of North America. The first was held at Northwestern University in 1893 and the second at the University of Toronto in 1924. International Congresses were held at intervals of approximately four years, except when war intervened, until 1936. There has been no international gathering of mathematicians since that time and it is the sincere hope of the Organizing Committee that the gathering in 1950 will be a truly international one, that the American mathematicians will attend in large numbers, and that all other countries will be well represented. The Council of the American Mathematical Society has voted unanimously to hold a Congress which will be open to mathematicians of all national and geographical groups.

Time and Place. The dates for the Congress have been fixed as August 30–September 6, 1950. Harvard University will be the principal host institution. A number of other institutions in metropolitan Boston will join in the entertainment of Congress visitors by arranging special features on their campuses.

Type of Congress. In recent years mathematicians have been much impressed by the success of the conference method for presenting recent research in fields where vigorous advances have just been made or are in progress. In view of the success of mathematical conferences on special topics which have been held in Russia, France and Switzerland and, more recently, at the Princeton Bicentennial Celebration, the 1950 Congress will include Conferences in several fields. For the 1940 Congress, Conferences in four fields had been planned. The number of Conferences was thus restricted lest the introduction of a promising and novel feature result in failure through the dissipation of interest and energy. A subcommittee of the Organizing Committee, under the chairmanship of Professor A. A. Albert, is now studying the question of the number and the fields of the Conferences to be included in the 1950 Congress and the results of the committee's deliberations will be reported at a later date.

Following the established custom, the Organizing Committee plans to have a number of invited hour addresses by outstanding mathematicians. In addition, sectional meetings for the presentation of contributed papers not included in Conference programs will be held in the following fields: I, Algebra and Theory of Numbers; II, Analysis; III, Geometry and Topology; IV, Probability and Statistics, Actuarial Science, Economics; V, Mathematical Physics and Applied Mathematics; VI, Logic and Philosophy, History and Education.

The official languages of the 1950 Congress will be English, French, German, Italian, and Russian.

Organization. The plans for the Congress are under the supervision of an Organizing Committee which was elected by the Council of the American Mathematical Society in February, 1948. The Chairman is Professor Garrett Birkhoff of Harvard University and the Vice Chairman is Professor W. T. Martin of Massachusetts Institute of Technology. Other members of the committee are: Professors J. L. Doob, G. C. Evans, J. R. Kline, Solomon Lefschetz, Saunders MacLane, Dean R. G. D. Richardson, Professors J. L. Synge, Oswald Veblen, J. L. Walsh, D. V. Widder, Norbert Wiener, and R. L. Wilder.

Many of the subventions promised for the 1940 Congress are still available. A Financial Committee under the chairmanship of Professor John von Neumann is endeavoring to secure additional funds. Besides support from Harvard University and Massachusetts Institute of Technology, generous subventions have been subscribed for the Congress by the Carnegie Corporation, the Institute for Advanced Study, the National Research Council, and the Rockefeller Foundation.

An Editorial Committee under the chairmanship of Professor Salomon Bochner will assume responsibility for the publication of the Proceedings of the Congress.

Professor J. R. Kline of the University of Pennsylvania has been named Secretary of the Congress and Dr. R. P. Boas, Executive Editor of Mathematical Reviews, has been designated Associate Secretary.

Entertainment. Harvard University has offered the use of its dormitories and dining rooms for mathematicians and their guests for the period of the Congress. The Organizing Committee hopes that it will be possible to furnish room and board without charge to all mathematicians from outside continental North America who are members of the Congress. Congress membership fees and rates for room and board will be announced well in advance of the opening of the Congress.

The Entertainment Committee, of which Professor L. H. Loomis of Harvard University is Chairman, is planning many interesting features, including a reception, garden party, symphony concert, and banquet. It is hoped that American mathematicians will be able to assist in the entertainment by putting their automobiles at the disposal of the Entertainment Committee for trips to be made out of Cambridge.

Every effort will be made to facilitate the travel at reasonable cost of foreign participants while in the United States. Previous to the Congress, opportunity will be given them to see New York City under the guidance of some mathematicians.

Information. Detailed information will be sent in due course to individual members of the American Mathematical Society and to foreign mathematical societies and academies. Others interested in receiving information may file their names in the Office of the Society, and such persons will receive from time

to time information regarding the program and arrangements.

Communications should be addressed to the American Mathematical Society, 531 West 116th Street, New York City 27, U.S.A.

STANFORD UNIVERSITY COMPETITIVE EXAMINATION

The third Stanford University Competitive Examination in Mathematics (see this MONTHLY, vol. LIII, 1946, pp. 406–409) was held April 10, 1948, in 53 high schools in California, with 206 students taking part. The following problems were proposed:

1. Consider the table:

$$\begin{aligned} 1 &= 1 \\ 2+3+4 &= 1+8 \\ 5+6+7+8+9 &= 8+27 \\ 10+11+12+13+14+15+16 &= 27+64 \end{aligned}$$

Guess the general law suggested by these examples, express it in suitable mathematical notation, and prove it.

2. Three numbers are in arithmetic progression, three other numbers in geometric progression. Adding the corresponding terms of these two progressions successively, we obtain

85, 76, and 84

respectively, adding all three terms of the arithmetic progression, we obtain 126. Find the terms of both progressions.

3. From the peak of a mountain, you see two points, A and B, in the plain. The lines of vision, directed to these points, include the angle γ. The inclination of the first line of vision to a horizontal plane is α, that of the second line β. It is known that the points A and B are on the same level and that the distance between them is c.

Express the elevation x of the peak above the common level of A and B in terms of the angles α, β, γ and the distance c.

4. A first sphere has the radius r_1. About this sphere circumscribe a regular tetrahedron. About this tetrahedron circumscribe a second sphere with radius r_2. About this second sphere circumscribe a cube. About this cube circumscribe a third sphere with radius r_3.

Find the ratios $r_1 : r_2 : r_3$ (which should be, according to Kepler, the ratios of the mean distances of the planets Mars, Jupiter, and Saturn from the Sun, but which are, in fact, rather different from the true ratios).

The writer of the best paper, C. L. Moller, Jr., Glendora, California, student at Citrus Union High School, Azusa, California, received a $500 scholarship at Stanford University. R. C. Hill, Los Angeles, California, student at Harvard School, North Hollywood, California, received honorable mention.

PERSONAL ITEMS

Professor C. V. Newsom, chairman of the Department of Mathematics of Oberlin College, has accepted the position of Assistant Commissioner for Higher Education for the State of New York. Editorial correspondence in regard to the MONTHLY should now be addressed to Dr. C. V. Newsom, State Education Building, Albany 1, New York.

Professor C. J. Coe represented the Association at the inauguration of President White of the University of Toledo, Ohio on May 11, 1948.

Professor L. R. Ford was a representative of the Association at the meeting of the American Council on Education, Chicago, Illinois on May 7–8, 1948.

Professor C. V. Newsom represented the Association at the inauguration of the president of Case Institute of Technology, Cleveland, Ohio on May 21, 1948.

Professor Charles Wexler was appointed to represent the Association at the inauguration of Dr. J. B. McCormick as president of the University of Arizona, Tucson, Arizona on May 5, 1948.

Professors George Pólya, E. B. Roessler, and R. M. Winger served as representatives of the Northern California Section of the Association at the Pacific Regional Conference on UNESCO on May 13–15, 1948.

Professor C. C. Chevalley of Princeton University, Assistant Professor Irving Kaplansky of the University of Chicago, and Associate Professor Norman Levinson of Massachusetts Institute of Technology have been awarded Guggenheim fellowships.

Dr. Churchill Eisenhart, chief of the Statistical Engineering Laboratory, National Bureau of Standards, has been elected a fellow of the Royal Statistical Society of Great Britain.

Associate Professor Maurice Ewing of Columbia University and Professor E. J. McShane of the University of Virginia have been elected to membership in the National Academy of Sciences.

Professor Gillie A. Larew, head of the Department of Mathematics at Randolph-Macon Woman's College, has been honored by the establishment of the Gillie A. Larew Chair of Mathematics. The fund for the endowment of the Chair was contributed by alumnae of the College.

Dean R. G. D. Richardson of Brown University has been awarded the honorary degree of Doctor of Laws by Brown University.

President W. M Whyburn of Texas Technological College has been awarded an honorary degree by Texas Technological College

Brown University announces the following promotions: Assistant Professors Herbert Federer and G. W. Whitehead to associate professorships, Lecturer Wouter van der Kulk to an assistant professorship, Dr. Bjarni Jonsson to an assistant professorship. Lectures were given at the Mathematics Colloquium of Brown University during the past year by the following mathematicians from abroad: Professor H. A. Bohr of the University of Copenhagen, Professor Henri Cartan of the University of Paris, and Professor Natan Aronszajn of the Centre National de la Recherche Scientifique.

Princeton University makes the following announcements: Associate Professor Alonzo Church has been promoted to a professorship; Assistant Professors R. H. Fox and J. W. Tukey have been promoted to associate professorships; Professor L. K. Hua of Tsing Hua University and Professor J. P. LaSalle have been appointed Lecturers; Associate Professor Witold Hurewicz of Massachusetts Institute of Technology has been appointed Visiting Professor.

Purdue University announces the following: Dr. Ralph Hull of Boeing Aircraft Corporation has been appointed Professor of Mathematics and Head of the Department of Mathematics replacing Dean W. L. Ayres, who will devote his full time to the School of Science; Dr. Olav Reiersol, now on a fellowship at the University of California, has been appointed Lecturer in Statistics; Assistant Professors M. E. Shanks and M. S. Webster have been promoted to associate professorships; Mr. Stanley Bolks, Mr. K. W. Crain, Mr. M. W. DeJonge, Mr. L. G. Black, Mr. P. W. Overman, and Dr. W. P. Reid have been promoted to assistant professorships.

Queens College announces the following promotions: Assistant Professor Arthur Sard to an associate professorship, and Associate Professors R. G. Archibald and John Williamson to professorships.

Rensselaer Polytechnic Institute announces the following promotions: Associate Professors R. E. Huston and W. G. Warnock to professorships, Assistant Professors J. D. Campbell, A. W. Jones, and Dis Maly to associate professorships, Mr. G. F. Guilford to an assistant professorship.

Syracuse University announces the appointments of Associate Professors M. F. Rosskopf and Atle Selberg and Assistant Professor H. E. Goheen. Assistant Professor D. E. Kibbey has been promoted to an associate professorship, Dr. P. W. Gilbert and Dr. Erik Hemmingsen have been promoted to assistant professorships.

The University of Alberta makes the following announcements: Professor E. W. Sheldon has retired with the title of Professor Emeritus; Professor J. W. Campbell has succeeded Dr. Sheldon in the position of Head of the Department of Mathematics; Mr. Edgar Phibbs and Mr. E. T. Sheffield have been appointed Assistant Professors; Mr. R. C. Jacka and Mr. Thorlief Fostvedt have been appointed Sessional Lecturers.

The University of Illinois announces the following appointments and promotions: Professor A. H. Taub has been appointed to a research professorship in applied mathematics; Mr. A. G. T. Carlton has been appointed to an assistant professorship; Associate Professors P. W. Ketchum and H. J. Miles have been promoted to professorships; Assistant Professors Harry Levy, C. W. Mendel and J. W. Peters have been promoted to associate professorships; Dr. V. R. Nuess has been promoted to an assistant professorship.

The University of Utah announces the following appointments: Professor C. R. Wylie Jr., chairman of the Department of Mathematics, Air Institute of Technology, Wright Field, has been appointed Professor and Chairman of the

Department of Mathematics; Dr. R. E. Chamberlin and Dr. James Wolfe have been appointed to assistant professorships.

Washington University makes the following announcements: Mr. Eric Kristensen of the University of Copenhagen has been appointed Lecturer; Dr. Zeev Nehari of Hebrew University has been appointed to an associate professorship; Assistant Professors H. M. Schaerf and G. B. Van Schaack have been promoted to associate professorships; Assistant Professor J. Y. Stephens has been promoted to an associate professorship of astronomy; Dr. Franklin Haimo and Mr. Marlow Sholander have been promoted to assistant professorships.

Wayne University announces the resignation of Associate Professor J. A. Pierce and the appointments of Associate Professor Benjamin Epstein and Assistant Professors Harvey Cohn and Gerald Harrison.

Dr. P. H. Anderson has been appointed Professor of Marketing at Loyola University.

Professor H. T. R. Aude of Colgate University has retired.

Dr. R. E. Basye, Agricultural and Mechanical College of Texas, has been promoted to an assistant professorship.

Associate Professor Z. W. Birnbaum of the University of Washington has accepted an appointment as director of the Laboratory of Statistical Research.

Assistant Professor Robert Breusch of Amherst College has been promoted to an associate professorship.

Professor W. H. Bussey of the University of Minnesota has retired.

Professor S. S. Cairns of Syracuse University has been appointed Head of the Department of Mathematics of the University of Illinois.

Professor B. H. Camp of Wesleyan University, Middleton, Connecticut, has retired.

Assistant Professor D. G. Chapman of the University of British Columbia has accepted a position as research assistant at the University of California.

Assistant Professor I. S. Cohen of the University of Pennsylvania has been appointed to an assistant professorship at Massachusetts Institute of Technology.

Assistant Professor Lester Dawson of Michigan College of Mining and Technology has been appointed to an associate professorship at Adams State College.

Dr. Wade Ellis has been appointed to an assistant professorship at Oberlin College.

Associate Professor H. F. Fehr of Montclair State Teachers College, New Jersey, has accepted a position as professor of mathematics at Teachers College, Columbia University.

Associate Professor W. W. Flexner of Cornell University has received an appointment as senior statistician in charge of transport statistics in the Department of Economic Affairs of the United Nations.

Professor F. C. Gentry of Louisiana Polytechnic Institute has been appointed to an associate professorship at Arizona State College.

Associate Professor S. G. Hacker of Washington State College has been promoted to a professorship.

Associate Professor D. A. Hatch of Lafayette College has retired.

Professor G. A. Hedlund of the University of Virginia has been appointed to a professorship at Yale University.

Assistant Professor D. M. Hester of Northwest Missouri State Teachers College has accepted a position at Baker University.

Associate Professor P. M. Hummel of the University of Alabama has been promoted to a professorship.

Mr. Sidney Kravitz of the Newark College of Engineering has received an appointment as mathematician with the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Professor D. C. Lewis of the University of Maryland has been appointed to a professorship at Johns Hopkins University.

Associate Professor E. R. Lorch of Columbia University has been promoted to a professorship.

Mr. R. C. Luippold of West Virginia Institute of Technology has been appointed to an assistant professorship at Antioch College.

Associate Professor C. T. McCormick of Illinois State Normal University has been promoted to a professorship.

Dr. N. W. McLachlan of London served as visiting professor of mathematics at Carnegie Institute of Technology during the second semester of 1947–1948.

Associate Professor A. K. Mitchell of the University of Maryland has accepted a position as aerodynamicist at the Applied Physics Laboratory of Johns Hopkins University.

Dr. Deane Montgomery has been appointed to permanent membership in the Institute for Advanced Study.

Mr. Dewey Moore has received an appointment as physicist with the National Advisory Committee for Aeronautics, Langley Field, Virginia.

Professor F. C. Ogg has been appointed Head of the Department of Mathematics of Bowling Green State University.

Mr. Richard Otter of Princeton University has been appointed to an assistant professorship at the University of Notre Dame.

Mr. Eugene Park, graduate assistant at the University of Wisconsin, has been appointed Assistant Professor at Clemson College.

Assistant Professor H. V. Price of the State University of Iowa has been promoted to an associate professorship.

Mr. A. O. Qualley of Lehigh University has accepted an appointment as assistant professor at Drake University.

Professor J. F. Randolph of Oberlin College has been appointed Head of the Department of Mathematics at the University of Rochester.

Professor W. D. Reeve of Teachers College, Columbia University has retired.

Dean R. G. D. Richardson of Brown University has retired.

Dr. Klaus Schocken, formerly of Siena College, is now a member of the Medical Department, Field Research Laboratory, Fort Knox, Kentucky.

Associate Professor K. C. Schraut has been promoted to a professorship at the University of Dayton.

Assistant Professor Ruth G. Simond of Morningside College has been appointed to an assistant professorship at the University of Vermont.

Associate Professor M. F. Smiley of Northwestern University has accepted an appointment as professor of mathematics at the State University of Iowa.

Assistant Professor E. R. Stabler has been promoted to an associate professorship at Hofstra College.

Assistant Professor D. W. Western of Brown University has been appointed to an associate professorship at Franklin and Marshall College.

Professor Anna Pell Wheeler of Bryn Mawr College has retired.

President W. M. Whyburn of Texas Technological College has been appointed to a professorship at the University of North Carolina.

Dr. Albert Wilansky of Brown University has been appointed Assistant Professor of Mathematics at Lehigh University.

Assistant Professor H. P. Wirth of the College of the City of New York has been promoted to an associate professorship.

The following appointments to instructorships are announced:

Case Institute of Applied Science: Mr. Charles Saltzer

Purdue University: Miss Marianne Bernstein, Mr. L. D. Kovach

Syracuse University: Mr. Helmut Aulback, Mr. S. G. Campbell, Mr. Eckford Cohen, Mr. Daniel Resch, Mr. C. J. Titus.

United States Naval Academy: Mr. F. P. Kowalewski, Jr.

University of Illinois: Dr. M. D. Springer

University of Utah: Mr. F. E. Haupt

Wayne University: Mr. Willard Clatworthy.

Connecticut College announces the appointment of Miss Elizabeth Hahnemann as teaching assistant.

Dr. Clara L. Bacon, professor emeritus of mathematics, Goucher College, died April 14, 1948. She was a charter member of the Association.

Dr. C. E. Comstock, professor emeritus of mathematics at Bradley University and a charter member of the Association, died on April 3, 1948.

Professor W. L. Johnson, head of the Department of Mathematics of Mississippi Southern College died March 14, 1948. He was Chairman of the Louisiana-Mississippi Section of the Association.

Professor Emeritus F. R. Sharpe of Cornell University died on May 18, 1948 at the age of seventy-eight years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following eighty-seven persons have been elected to membership on applications duly certified:

R. H. ACKERSON, A.M.(Columbia) Asst. Professor, Catawba College, Salisbury, N. C.

R. V. ANDERSON, A.M.(Colorado) Instructor, Colorado Agricultural & Mechanical College, Fort Collins, Colo.

P. M. BAILEY, B.S.(Wyoming) Student, State University of Iowa, Iowa City, Iowa

MARY M. BAXTER, M.A.(Columbia) Instructor, Junior College of Kansas City, Mo.

LULU BECHTOLSHEIM, Ph.D.(Zurich) Asst. Professor, University of Redlands, Calif.

MARTIN BERMAN, M.S.(Virginia Polytechnic Institute) Instructor, University of Cincinnati, Ohio

L. J. BERNER, B.A.(St. Norbert College) Instructor, St. Norbert College, West De Pere, Wis.

W. J. BRUNS, Ph.D.(Strassburg) Visiting Professor, Syracuse University, N. Y.

BREDIE E. (Mrs. Buron) BUFFKIN, A.B.(Coker) Instructor, University of South Carolina, Columbia, S. C.

P. B. BURCHAM, Ph.D.(Northwestern) Asst. Professor, University of Missouri, Columbia, Mo.

MARY P. BURKART, B.S.(Nazareth College) Graduate Asst., University of Detroit, Mich.

C. D. CALHOON, M.A.(Illinois) Asst. Professor, University of Toledo, Ohio

P. A. CLEMENT, M.A.(State College of Washington) Teaching Asst., University of California at Los Angeles, Calif.

W. J. CONNER, M.A.(Texas) Asst. Professor, The Citadel, Charleston, S. C.

GRACE CUTLER, M.A.(Toledo) Asst. Professor, University of Toledo, Ohio

R. C. DAVIS, B.S.(Akron) Instructor, University of Akron, Ohio

DOUGLAS DERRY, Ph.D.(Gottingen) Asso. Professor, University of British Columbia, Vancouver, B. C.

J. G. DIAZ, Agricultural Engineer, Alarcon 27, Madrid, Spain

I. A. DODES, Ph.D.(New York University) Teacher, Stuyvesant High School, 345 East 15th St., New York, N. Y.

MARY C. DOREMUS, M.A.(Columbia) Teacher, North High School, Denver, Colo.

HELEN ENGEBRETSON, M.A.(Minnesota) Instructor, State College, Brookings, S. D.

O. J. FALKENSTERN, B.S.(Montana State) Instructor, Colorado Agricultural & Mechanical College, Fort Collins, Colo.

CLEOTA G. FRY, Ph.D.(Purdue) Asst. Professor, Purdue University, Lafayette, Ind.

R. N. GOSS, M.S.(Iowa State) Graduate Asst., Iowa State College, Ames, Iowa

LILLIAN GOUGH, B.A.(Buffalo) Instructor, University of Buffalo, N. Y.

J. B. GREELEY, M.S.(Massachusetts Institute of Technology) Head of Department, Utica College, N. Y.

MARSHALL HALL, JR. Ph.D.(Yale) Asso. Professor, Ohio State University, Columbus, Ohio

LUCILLE F. HETZELT, A.M.(Syracuse) Instructor, Syracuse University, N. Y.

WILBUR HIBBARD, M.A.(Columbia) Instructor, Lehigh University, Bethlehem, Pa.

CLARICE HOBENSACK, M.A. (Ohio State) Teacher, Cincinnati Public Schools, Ohio

V. A. HOYLE, Ph.D.(Princeton) Professor, University of North Carolina, Chapel Hill, N. C.

R. S. JACOBSEN, M.S.(Iowa State) Asso. Professor, Luther College, Decorah, Iowa

M. E. JENKINS, M.S.(Syracuse) Asst. Professor, Mohawk College, Utica, N. Y.

CHOSABURO KATO, Ph.D.(Ohio State) Asso. Professor, Denison University, Granville, Ohio

G. L. KEPPERS, M.A. (Colorado State College of Education) Instructor, Iowa State Teachers College, Cedar Falls, Iowa

H. L. KINSOLVING, A.M.(Harvard) Asst. Professor, U. S. Naval Academy, Annapolis, Md.

CHARLES KOREN, M.A.(Columbia) Chairman, Math. Department, Bayonne Junior College, N. J.

F. W. LANE, M.S.(St. Bonaventure) Asst. Professor, Mohawk College, Utica, N. Y.

L. S. LAWS, A.M.(Stanford) Asst. Professor, University of Minnesota, Minneapolis, Minn.

TA CHUNG-HENG LI, Ph.D.(Munich) Asst. Professor, Drake University, Des Moines, Iowa

J. J. LIVERS, Ph.D.(Michigan) Professor, Montana State College, Bozeman, Mont.

B. H. LOWE, M.S.(Colorado) Instructor, University of Akron, Ohio

NATHANIEL MACON, M.A.(North Carolina) Teaching Fellow, University of North Carolina, Chapel Hill, N. C.

POMPEY MAINARDI, M.A.(Montclair) Asso. Professor, Newark College of Engineering, N. J.

C. I. MALE, M.C.E.(Union) Asso. Professor, Union College, Schenectady, N. Y.

DIS MALY, M.A.(Harvard) Asst. Professor, Renasselaer Polytechnic Institute, Troy, N. Y.

W. H. MARLOW, B.S.(St. Ambrose College) Graduate Asst., State University of Iowa Iowa City, Iowa

F. J. MARQUIS, M.A.(Miami) Asst. Professor, University of Toledo, Ohio

J. M. MARR, B.S.(Central Missouri State College) Asst. Instructor, University of Missouri, Columbia, Mo.

H. C. MAYER, JR. M.S.(Iowa) Instructor, University of Idaho, Moscow, Idaho

BETTY MCKNIGHT, M.A.(Southern Methodist University) Instructor, Centenary College, Shreveport, La.

E. K. MCLACHLAN, M.A.(Texas) Instructor, Baylor University, Waco, Texas

E. J. MILTENBERGER, M.A.(Miami) Asst. Professor, Miami University, Oxford, Ohio

M. R. MOORE, S.M.(Massachusetts Institute of Technology) Graduate Asst., Bowling Green State University, Ohio

J. B. NELSON, A.M.(University of Southern California) Instructor, Los Angeles City College, Calif.

P. W. OVERMAN, M.S.(Indiana) Asst. Professor, Purdue University, Lafayette, Ind.

J. F. PARIS, Student, George Pepperdine College, Los Angeles, Calif.

C. R. PARTINGTON, A.B.(Earlham) Instructor, Purdue University, Lafayette, Ind.

R. L. PRUITT, M.S.(Atlanta) Asst. Professor, Albany State College, Ga.

R. C. RAND, Ph.D.(Maryland) Instructor, U.S. Naval Academy, Annapolis, Md.

T. R. RICHARDS, M.S.(Bucknell) Asst. Professor, Wilkes College, Wilkes-Barre, Pa.

C. L. RIGGS, M.A.(Michigan) Instructor, University of Kentucky, Lexington, Ky.

ABRAHAM ROSENFELD, M.S.(Massachusetts Institute of Technology) Asst. Professor, University of Massachusetts, Ft. Devens, Mass.

J. W. SAWYER, M.A.(Missouri) Instructor, University of Missouri, Columbia, Mo.

JEANETTE, H. SHERWOOD, M.A.(Michigan) Instructor, University of Detroit, Mich.

R. W. SHOEMAKER, M.S.(Toledo) Asst. Professor, University of Toledo, Ohio

SISTER M. DOROTHEA, S.S.J., M.A.(Cornell) Instructor, Nazareth College, Rochester, N. Y.

SISTER MARY RAPHAEL HAFNER, A.B.(College of Chesnut Hill) Instructor, Immaculata College, Pa.

J. L. SLECHTICKY, M.S.(Washington University) Asst. Professor, Toledo University, Ohio

C. B. SMITH, Ph.D.(Wisconsin) Asso. Professor, University of Florida, Gainesville, Fla.

J. E. SNOVER, B.A.(New York State College for Teachers) Instructor, Mohawk College, Utica, N. Y.

R. H. SPOHN, A.B.(Lebanon Valley College) Instructor, Lehigh University, Bethlehem, Pa.

E. M. STEINBACH, A.B.(Northern State Teachers College) Instructor, University of Detroit, Mich.

J. R. SULLIVAN, A.B.(Georgetown) Instructor, Clemson College, S. C.

J. D. SWIFT, Ph.D.(California Institute of Technology) Instructor, University of California at Los Angeles, Calif.

ETHEL W. TAPPER, B.S.(Illinois) Librarian and Asst. Professor, Aurora College, Illinois

C. J. TEGELS, B.A.(St. John's University) Instructor, University of Detroit, Mich.

W. J. TUCKER, Asst. Projects Engineer, Sperry's Radio Engineering Research, New York, N. Y.

NURA D. TURNER, M.A.(State University of Iowa) Instructor, New York State College for Teachers, Albany, N. Y.

G. W. TYLER, M.A.(Duke) Asso. Professor, Virginia Polytechnic Institute, Blacksburg, Va.

C. B. WALTON, B.S.(Pennsylvania) Senior Engineer, Federal Power Commission, Washington, D. C.

M. D. WEINERT, M.Ed.(Boston Teachers College) Instructor, University of Massachusetts, Ft. Devens, Mass.

MARGARET F. WILLERDING, Ph.D.(St. Louis) Instructor, Harris Teachers College, St. Louis, Mo.

A. D. WIRSHUP, M.A.(Columbia) Instructor, Multnomah College, Portland, Ore.

W. D. WOOD, B.S.(Case Institute of Technology) Graduate Student, Oberlin College, Ohio

A. G. WOOTTON, B.S.(Rutgers) Asst. Professor, Mohawk College, Utica, N. Y.

V. A. ZORA, B.S.(Pittsburgh) Chemical Process Engineer, Blaw-Knox Const. Co., Chemical Plants Division, Pittsburgh, Pa.

THE EIGHTH CARUS MATHEMATICAL MONOGRAPH "RINGS AND IDEALS" BY N. H. McCOY

This worthy addition to the series of Carus Monographs is a clear and concise exposition of the fundamental concepts and results in the elementary theory of rings, with some emphasis on the role of ideals in the theory.

Members of the Association may purchase one copy each of this monograph for $1.25. Orders accompanied by remittance should be sent to: The Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Additional copies for members and copies for non-members are priced at $2 each and must be purchased directly from the Open Court Publishing Company, La Salle, Illinois.

It is expected that the prices of all Carus Monographs will be increased as of December 1, 1948, and the above-mentioned prices will hold only until that date.

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1948 by vote of the membership of the Association in the Sections indicated:

Allegheny Mountain	H. L. Dorwart, Washington and Jefferson College
Indiana	P. D. Edwards, Ball State Teachers College
Kentucky	M. C. Brown, University of Kentucky
Metropolitan New York	F. H. Miller, Cooper Union
Nebraska	C. C. Camp, University of Nebraska
Northern California	F. R. Morris, Fresno State College
Oklahoma	N. A. Court, University of Oklahoma
Rocky Mountain	A. J. Lewis, University of Denver
Wisconsin	R. H. Bardell, University of Wisconsin in Milwaukee

Since the system of Regional Governors is being replaced by a system of

Sectional Governors, no elections have been held to replace the Regional Governors whose terms expired on July 1, 1948.

BACK NUMBERS WANTED

One dollar a copy will be paid (or credited on members' dues) for any of the following issues of this MONTHLY until fifty copies of each number are received:
1946—January, May, June-July, August-September
1947—January, December
1948—January, February, May

Copies should be mailed to the Mathematical Association of America, University of Buffalo, Buffalo 14, New York with the name and address of the sender marked clearly on the package.

JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The tenth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California on Saturday, January 24, 1948. Professor George Pólya, Chairman of the Section, and Professor G. C. Evans, Vice-Chairman, presided at the morning session, and Mr. S. A. Francis presided at the afternoon session.

The attendance was ninety-three including the following twenty-nine members of the Association: H. M. Bacon, G. A. Baker, T. J. Bass, E. M. Beesley, M. T. Bird, A. C. Burdette, Louise Chin, M. A. Dernham, G. C. Evans, S. A. Francis, C. M. Fulton, J. M. Good, W. R. Hanson, Sophia McDonald, E. D. Miller, F. R. Morris, W. H. Myers, C. D. Olds, George Pólya, Edris P. Rahn, R. M. Robinson, E. B. Roessler, Kathryn Rolfe, Abraham Seidenberg, Pauline Sperry, Ruth Sumner, Gabor Szegö, L. A. Walker, A. R. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, G. C. Evans, University of California; Vice-Chairman, H. M. Bacon, Stanford University; Secretary-Treasurer, E. B. Roessler, University of California at Davis; Representative on the *California Journal of Secondary Education*, Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor Harold Davenport of University College, London, and Visiting Professor at Stanford University, gave an address during the morning session.

The following papers were read:

1. *A matrix arising in correlation theory*, by Professor H. M. Bacon, Stanford University.

Consider a set of n variables $x_1, x_2, x_3, \cdots, x_n$, representing observations taken in n successive years, with correlation coefficients r_{ij} between x_i and x_j. If the correlation coefficients are assumed to diminish in such a way that $r_{ij}=r_{ij}=1-|i-j|p$, where $0<p<2/(n-1)$, then the determinant $|r_{ij}|=R$ can be expressed in terms of n and p. It was shown that

$$R = 2^{n-1}p^{n-1}[2 - (n - 1)p].$$

Expressions for the cofactors R_{ij} of the elements r_{ij} of R were given. Assuming the x_i to be normally distributed, an expression for the frequency function was obtained in terms of n and p.

2. *Treatment of trigonometric functions in calculus*, by Dr. C. M. Fulton, University of California at Davis.

Instead of introducing the number e as a limit, one can define $\log x = \int_1^x (1/t)dt$ for $x>0$, or taking its existence for granted, the function $\log x$ can be characterized by means of the properties $D_x \log x = 1/x$, $\log 1 = 0$. The addition theorem is proved by observing that $D_x \log ax = D_x \log x$. Further properties of the logarithm and its inverse function follow easily. A graphical construction of the exponential curve using pieces of tangent lines is recommended.

In an analogous way let $\tan^{-1} x = \int_0^x [1/(1+t^2)]\,dt$. From $D_x \tan^{-1}[(a+x)/(1-ax)] = D_x \tan^{-1} x$ the addition formula of the inverse tangent is found. Some difficulties arise with respect to the intervals. The inverse function and the other trigonometric functions and their basic properties can be derived by simple manipulations. A geometric interpretation of the defining equation for $\tan^{-1} x$ is given. Finally it is pointed out that $\tanh^{-1} x$ can be handled in like fashion.

3. *The geometry of numbers*, by Professor Harold Davenport, University College, London, and Visiting Professor at Stanford University, introduced by the Chairman.

In the geometry of numbers, we treat a general class of problems in the theory of numbers by methods which are suggested by a geometrical interpretation. The problems in question relate to "Diophantine inequalities," *i.e.* inequalities which are to be satisfied by integral values of the variables. Questions of this kind arise quite naturally. For example, if we wish to prove the theorem of Fermat that every prime p of the form $4n+1$ is representable as x^2+y^2, we easily find a general formula which represents only multiples of p as x^2+y^2, and the proof can be completed by showing that the formula also represents a number less than $2p$ (and not zero).

The geometrical approach is that of conceiving an inequality $f(x_1, \cdots, x_n) < \lambda$ as representing a certain region in n-dimensional space. We then ask for conditions which will ensure that this region contains a point with integral coordinates. If the inequality is trivially satisfied by $x_1 = \cdots = x_n = 0$, we then need a point other than the origin.

In the special example where the inequality represents an ellipse in the plane, it is not difficult to prove that if the area of the ellipse is greater than $2\pi/\sqrt{3}$, there is always a point with integral coordinates in it, other than the origin. But the argument used is too special to be applied effectively in n dimensional space, or to more general regions.

A simple train of reasoning was discovered by Minkowski (about 1891) which led him to this very general theorem: Any body in n dimensional space, which is convex and symmetrical about the origin, and which has a volume greater than 2^n, must contain a point other than the origin with integral coordinates. This result has many applications in the theory of numbers.

In the last few years, new methods have been developed by several English mathematicians for dealing with problems in which the region represented by the inequality is not convex.

4. *How to solve it*, (Report on the New Haven Symposium of the Association) by Professor George Pólya, Stanford University.

Reports of the meeting appeared in this MONTHLY, vol. 54, p. 612, and vol. 55, p. 22.

5. *A problem in calculus*, by Professor M. T. Bird, San Jose State College.

Given a continuous curve $y=f(x)$ which intersects the line $y=mx+b$ in just two points, namely the points for which $x=a$ and $x=c$. It is required to find the volume generated when the area between the curve and the line is revolved about the line. An erroneous analysis with elements of volume which are circular plates is seen to lead to the correct value for the volume. An analysis with elements of volume which are conical shells is seen to be appropriate.

6. *The variance of the proportions of samples falling within a fixed interval for a normal population*, by Professor G. A. Baker, University of California at Davis.

The usual variance of the proportions of a sample falling in a fixed interval given in elementary textbooks is the binomial variance $p(1-p)/N$, and is the same for all parent populations. If a normal population is sampled, estimates of $\bar{x}$ and s^2 being formed in the usual way, and if p is estimated by putting $\bar{x}$ and s^2 for m and σ^2 in the integral defining p, we get an estimate of p that has a considerably smaller variance than the corresponding binomial variance.

7. *The Banach-Tarski paradox*, by Professor R. M. Robinson, University of California.

It is possible to cut a solid sphere into five pieces and reassemble then by rigid motions to form two solid spheres of the same size. No fewer than five pieces may be used, but one of the pieces may consist of just one point. The proof of this result will appear in *Fundamenta Mathematicae*. A similar result with an unspecified finite number of pieces was proved by Banach and Tarski in 1924. It is evident from the Banach-Tarski paradox that there cannot be any completely general method of defining volume so that it will be additive and invariant under motion.

E. B. ROESSLER, *Secretary*

FEBRUARY MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Tulsa on Friday, February 13, 1948. Professor D. R. Shreve, Chairman of the Section, presided.

Fifty-three persons attended the meeting, including the following sixteen members of the Association: A. Bernhart, J. C. Brixey, H. N. Carter, N. A. Court, N. A. Eisen, J. O. Hassler, O. H. Hamilton, J. E. LaFon, Eunice Lewis, R. D. McDole, Josephine Mitchell, E. P. Northrop, C. C. Pruitt, D. R. Shreve, R. W. Veatch, Marion West.

At the business session the following officers were elected: Chairman, J. E. LaFon, University of Oklahoma; Vice-Chairman, H. V. Huneke, Northwestern State College; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following nine papers:

1. *Professor W. T. Short Memorial*, by Professor N. A. Court, University of Oklahoma.

2. *How to use the history of mathematics in teaching*, by Professor J. O. Hassler, University of Oklahoma.

Professor Hassler recounted briefly the most significant items in the history of the development of algebra, trigonometry, analytics, and calculus, suggesting how and where the material could be included with the instruction in the subject matter without the necessity of a separate course in the history of mathematics. He also made a plea for the inclusion of more historical material in mathematical instruction in order to give the students the proper appreciation of the place of mathematics in the development of civilization.

3. *Use of trigonometric and hyperbolic functions in empirical equations*, by N. A. Eisen, Mid-Continent Petroleum Corporation.

A procedure for fitting an equation of the form $y=a+bf(x)$, where $f(x)$ is a trigonometric or hyperbolic function, was developed. This included the determination of the range of the angle to be used. The procedure involved the determination of four constants.

4. *Mathematics of the drafting board*, by Mr. S. S. Orman, Tulsa Industrial Arts High School, introduced by Professor D. R. Shreve.

The speaker described the geometrical constructions and calculations pertaining to drafting, with emphasis on the need to visualize three-dimensional figures. The comprehension and accuracy necessary to do detail design work was discussed.

5. *On asymptotes*, by Professor D. R. Shreve, University of Tulsa.

The vertical asymptotes of the locus of an irreducible integral rational equation of the form $a_0x^n+a_1x^{n-1}+\cdots+a_n=0$ are known to be the locus of $a_0=0$. (Here a_0 is understood to be a polynomial in y of degree $m\geqq 1$.) An extension to the determination of asymptotic planes and cylinders of surfaces was indicated.

6. *Summability of double orthogonal series whose coefficients satisfy certain conditions*, by Professor Josephine Mitchell, Oklahoma A. and M. College.

A discussion of the known results on convergence and $(C, 1)$ summability of simple orthogonal series was given. A theorem on the connection between $(C, 1, 1)$ summability and the convergence of the double orthogonal series which has been proved by the author was described.

7. *An upper bound for the number of irreducible factors of a polynomial with coefficients in a Gaussian domain*, by Mr. R. R. Gordon, introduced by Professor J. C. Brixey.

Given a Gaussian domain G, and k distinct elements $b_j\in G$, $j=1, 2, \cdots, k$, and a polynomial of degree n

$$P(x)\equiv\prod_{i=1}^{f}p_i(x) \tag{1}$$

where the coefficients of $P(x)$ and each $p_i(x)$ are in G, each $p_i(x)$ being of degree greater than zero, and irreducible into polynomials of type (1). Then the total number of irreducible factors f in (1) for any positive rational integral k and n is such that

$$f\leqq\frac{un}{k}+\frac{1}{k}\sum_{j=1}^{k}c[P(b_j)] \tag{2}$$

where u is the number of distinct units in G, and for any element $b\in G$, $c[P(b_j)]$ is the total number of primes in G contained by any factored form of b in which each factor is a unit or a prime in G. By assigning certain conditions to (2) and some of its arrangements, certain interesting conclusions can be obtained.

8. *Roots of polynomials*, by Professor A. Bernhart, University of Oklahoma

In applying Descartes' rule of signs, a negative coefficient between two positive coefficients of greater absolute value indicates a pair of conjugate imaginary roots.

The *average* product of roots taken r at a time is the same for a polynomial and its derivative. Representing the roots of a cubic by A, B, C, on a Gauss plane, the quadratic roots L, N may be interpreted geometrically by means of the auxiliary points R, T at the centroids of equilateral triangles with base AB.

Let S be the midpoint of AB and RT, and M the midpoint of LN. Then SR and ST are the geometric means of SL and SN; and conversely ML and MN are the geometric means of MR and MT. Thus L, N, R, T are concyclic, and LN is parallel to an asymptote of the equilateral hyperbola with center S which passes through R.

9. *Some generalizations of Brouwer's fixed point theorem for n-cells*, by Professor O. H. Hamilton, Oklahoma A. and M. College.

Brouwer's fixed point theorem for n-cells was stated. Generalizations of the theorem were classified as those which deal with continua more general than an n-cell subjected to a continuous transformation, and those which pertain to transformations more general than single valued continuous transformations. Examples of applications of fixed point theorems to analysis were given.

J. C. BRIXEY, *Secretary*

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-eighth regular meeting of the Southern California Section ofthe Mathematical Association of America was held at the University of Redlands, Redlands, California, on Saturday, March 13, 1948. Professor D. V. Steed Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was one hundred, including the following forty-eight members of the Association: L. J. Adams, O. W. Albert, H. M. Bacon, E. F. Beckenbach, May M. Beenken, Clifford Bell, L. T. Black, Herbert Buseman, F. A. Butter, Jr., W. D. Cairns, Frances L. Campbell, L. M. Coffin, E. L. Crow, D. R. Curtiss, P. H. Daus, R. P. Dilworth, Iva B. Ernsberger, B. K. Gold, J. W. Green, H. J. Hamilton, W. L. Hart, R. B. Herrera, M. R. Hestenes, P. G. Hoel, R. E. Horton, D. H. Hyers, C. G. Jaeger, B. W. Jones, Margaret B. Lehman, Ada McClellan, Sophia L. McDonald, G. F. McEwen, F. R. Morris, P. M. Niersbach, C. L. Olsen, W. T. Puckett, H. R. Pyle, E. C. Rex, J. M. Robb, G. E. F. Sherwood, Ernst Snapper, I. S. Sokolnikoff, R. H. Sorgenfrey, D. V. Steed, C. W. Trigg, S. E. Urner, Morgan Ward, Euphemia R. Worthington.

At the business meeting the following officers were elected for the next academic year: Chairman E. F. Beckenbach, University of California at Los Angeles; Vice-Chairman, H. R. Pyle, Whittier College; Program Committee, Herbert Buseman (Chairman), F. A. Butter Jr., E. J. Hills, and the Secretary, P. H. Daus, ex-officio. The next meeting was scheduled for March 12, 1949 at John Muir Junior College, Pasadena, California.

The following papers were presented:

1. *A college teacher's attitude toward the secondary curriculum in mathematics*, by Professor W. L. Hart, University of Minnesota.

Professor Hart called attention to the fact that uncomplimentary opinions are frequently expressed about the attitudes of college teachers of mathematics toward the curriculum in secondary mathematics. He indicated his belief that many of these adverse opinions, and certain serious evils in the secondary field, are a consequence of attaching the label "college preparatory" to the high school courses containing the most substantial mathematics. Then, for a college teacher of mathematics, Professor Hart advised a viewpoint designed to be helpful to the secondary field as well as to the colleges.

2. *Current practices and problems in secondary mathematics*, by Mr. Dale Carpentar, Supervisor, Mathematics Education Section, Los Angeles City Schools, introduced by Professor P. H. Daus.

Mr. Carpentar reviewed the history of the reorganization or reconstruction of high school mathematics, and the effect that the organization of Junior High Schools has had upon the problem. He suggested a mathematics program which would meet the vocational needs and aptitudes of all major groups of secondary students, and gave reasons why progress toward such a program

has been slow. He cited what the Los Angeles School District is planning to do to meet this problem. This involves the organization of a non-college preparatory sequence in mathematics, and an experimental mathematics program for college preparatory students who are not science majors. This experiment, which is still in the organizational stage, represents a cooperative endeavor of the Los Angeles City and County High Schools and the University of California.

3. *Geometry and physical space*, by Professor H. P. Robertson, California Institute of Technology, introduced by Professor J. W. Green.

This was an invited address given before a joint meeting of this Section and the Southern California Section of the American Association of Physics Teachers.

The speaker discussed the relation between the notion of curvature of a space, with its geometrical implications, and the nature of the device with which distance in the space is measured. This relation was illustrated by consideration of a plane in which exists a steady temperature distribution, and in which distances are measured by a rod which expands with change of temperature according to a linear law. The geometry which one would deduce from such measurements is a non-euclidean geometry, which in the case of a linear temperature distribution is the geometry of the Poincare half-plane.

4. *Report of the joint committee on mathematical education*, by Professor F. R. Morris, Fresno State College.

This was a progress report on the activities of the committee, appointed last year, and representing the Northern California and Southern California Sections of the Association.

5. *Concerning variations on Newton's method*, by Professor H. J. Hamilton, Pomona College.

Let z^* be a regular point for $f(z)$ and $\phi(z)$, and a simple zero for $\phi(z)$. Putting $z_n=f(z_{n-1})$, for $n=1, 2, \cdots$, and taking z_0 sufficiently near to z^*, we shall have $z_n\to z^*$ with $(z_n-z^*)=0[(z_{n-1}-z^*)^k]$, ($k$ an arbitrary integer ≥ 2), if and only if $f(z)$ is expressed in certain ways in terms of $\phi(z)$ and its derivatives. Thus for $k=2$ we have $f(z)=z-[\phi(z)/\phi'(z)]+h(z)[\phi(z)]^2$, where $h(z)$ is regular at z^*. The derivations involve elementary manipulations of power series.

6. *Problems with many or no solutions*, by Professor Herbert Buseman, University of Southern California.

The curves of constant width furnish a solution of a mechanical problem which at first sight seems to have the circle as the only solution. The problem of Kakaya to find a domain of least area in which a segment ab can be moved continuously such that a falls on b, and b on a, was discussed as an example of a seemingly reasonable problem with no solution.

7. *Calculations of some properties of certain supersonic wing sections*, by Dr. F. A. Butter, Jr., Hughes Aircraft Co.

A method is derived for the calculation of the area, the principal and polar moments of inertia, and the squares of the corresponding radii of gyration of a biconvex supersonic airfoil section in the form of a hollow double circular segment with chord c, thickness t, and constant skin thickness τ(=difference between internal and external radii). Let Q denote any one of the seven quantities to be calculated. The calculation of Q is effected by two divisions, three to five multiplications, and the reading of a value from curves. The error due to curve interpolation does not exceed 0.8 percent, which accuracy is more than ample for the purpose intended, that is, the preliminary design phase of an airfoil. If Q has the dimensions L^s, where L is length, and $s=2$ or 4, then Q is expressed in the form

$$Q = N\tau^a c^b t^c H(g, k)$$

where N (=1 or 2) is dimensionless, $g=2r/t$=skin ratio, $k=t/c$=thickness ratio, and α, b, and a are non-negative integers with $\alpha+b+a=s$. The "shape function" $H(g, k)$ has positive lower and upper bounds in the closed unit square $0\leqq g\leqq 1$, $0\leqq k\leqq 1$. Each of the seven functions $H(g, k)$ has been calculated to five significant figures for g and k equal to 0.0, 01., $\cdots$, 1.0, and is presented graphically as a function of g with one curve for each value of k, all plotted on the same pair of axes.

8. *On p-adic numbers*, by Professor B. W. Jones, Cornell University.

The p-adic numbers were defined in terms of formal series, and their fundamental properties derived. Their relation to congruence theory was discussed, and some applications were briefly dealt with.

P. H. DAUS, *Secretary*

MARCH MEETING OF THE PACIFIC NORTHWEST SECTION

The second annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the University of Oregon, Eugene, on Friday and Saturday, March 26–27, 1948.

Fifty-nine persons attended, including the following forty-four members of the Association: W. H. Bunch, L. G. Butler, Paul Civin, C. L. Clark, Douglas Derry, N. S. Free, K. S. Ghent, W. M. Gilbert, F. L. Griffin, S. G. Hacker, H. H. Irwin, R. D. James, J. M. Kingston, W. J. Kirkham, Celia E. Klotz, M. S. Knebelman, J. C. R. Li, J. J. Livers, C. F. Luther, L. H. McFarlan, H. C. Mayer, A. S. Merrill, W. E. Milne, A. F. Moursund, D. C. Murdoch, Ivan Niven, Andrewa Noble, T. G. Ostrom, T. S. Peterson, A. R. Poole, H. F. Price, Margaret Ramsey, R. A. Rosenbaum, J. J. Rowland, R. B. Saunders, W. G. Scobert, R. H. Stair, W. M. Stone, G. H. Van Arkel, J. R. Vatnsdal, G. A. Williams, L. B. Williams, R. M. Winger, F. E. Wood.

At the business meeting of the Section the following officers were elected for the year 1948–9: Chairman, W. E. Milne, Oregon State College; Vice-Chairman, R. M. Winger, University of Washington; Secretary-Treasurer, S. G. Hacker, State College of Washington. The section voted to accept the invitation of Oregon State College to hold the next annual meeting in Corvallis in the spring of 1949.

The Friday afternoon session was opened by Professor M. S. Knebelman, Chairman of the Section, and Professors F. L. Griffin and R. M. Winger were invited by him to preside. Professor Ivan Niven, University of Oregon, delivered an invited hour address on *The Density of a Set of Integers*. He gave various definitions of number theoretic density and some of the basic theorems in each case, including a discussion of the recent work of R. C. Buck on measure theoretic density and H. B. Mann's solution of the $\alpha+\beta$ problem.

Following the business meeting on Friday night a conference on the mathematical training of prospective secondary school teachers in the Pacific Northwest was held, with Professor Milne presiding. Professor Knebelman presented a brief report of the studies being made by this Association's Committee for the Coordination of Studies in Mathematical Education, of which he is a member. As a result of the spirited discussion which the conference provoked, it was de-

cided to appoint a committee with representation from the several institutions represented in this section to study this important matter and to make specific recommendations, including the possible specification of minimum requirements in mathematical preparation for certification of prospective high school teachers of mathematics in this region. Professor H. H. Irwin, State College of Washington, was appointed chairman of this committee.

The first five of the following papers as well as Professor Niven's address were presented at the Friday afternoon session. The remainder were read on Saturday morning, Professors A. S. Merrill and R. D. James presiding at the invitation of Professor Milne.

1. *Some theorems on measure*, by Professor R. A. Rosenbaum, Reed College.

A generalization of a theorem of Ostrowski on the Lebesgue measure of the direct sum of certain euclidean sets was proved. This theorem, together with some others, was applied to a discussion of finite solutions of sub-additive functions.

2. *Power series and Laplace transforms with multiply monotonic conditions*, by Mrs. Helen Clucas, University of Oregon, introduced by Professor Paul Civin.

The behavior of a power series with multiple monotonic functions was discussed. In particular, asymptotic evaluations were given for the remainder term of power series with coefficients of monotonic order 3 and of Laplace transforms of functions of monotonic order 4. (In the absence of the author this paper was read by Mr. W. G. Scobert, University of Oregon.)

3. *The solution of linear integral equations by means of Wiener integrals*, by Professor T. G. Ostrom, Montana State University.

An abstract of this paper has appeared in the *Bull. Amer. Math. Soc.*, vol. 53, 1947, p. 490 (Abstract 53-5-222).

4. *The spherical representation of a hypersurface*, by Professor M. S. Knebelman, State College of Washington.

This paper dealt primarily with the total curvature of a hypersurface expressed in terms of the curvature tensor. Allendoerfer gave such an expression as a rational function for a subspace of an even number of dimensions. The expression obtained in this paper is valid for any number of dimensions but it is not rational, and in the case of a space of an odd number of dimensions it may fail to be real.

5. *The June meeting of the American Mathematical Society at the University of British Columbia*, by Professor R. D. James, University of British Columbia.

Professor James gave an interesting account of the extensive scientific and social preparations which have been made for the entertainment of the mathematicians and other scientists meeting in Vancouver during the month of June .A cordial invitation was extended to the members of this section to attend these meetings and also to enjoy the many recreational facilities of Vancouver and its environs.

6. *On the precision of representative sampling*, by Professor Z. W. Birnbaum, University of Washington.

The frequently stated belief that the precision of the mean of a representative sample is always at least as good as that of the mean of a random sample of the same size is shown to be incorrect

for sampling without replacements. This paper was read by title.

7. *A discussion of the basic course in statistics*, by Professor W. J. Dixon, University of Oregon, introduced by Professor A. F. Moursund.

An outline and discussion of a course in statistics for all students which follows the recommendations of the National Research Council were presented.

8. *The quadrilaterals of Pascal's hexagram*, by Mr. W. H. Bunch, Nyssa (Oregon) High School.

This paper has since been published in this MONTHLY, vol. 55, 1948, p. 210.

9. *An intermediate algebra course for third year*, by Professor Douglas Derry, University of British Columbia.

The author criticized the traditional theory of equations course as failing to be in accord with modern algebraic development. An alternative course in linear algebra and analytic geometry was suggested.

10. *The axiomatic method in analysis*, by Professor Paul Civin, University of Oregon.

Certain weaknesses are pointed out in the logical structure of the analysis program as it is typically presented to the undergraduate. The desirability and feasibility of placing analysis on the same foundation as algebra or topology at the beginning graduate level is noted. This paper was read by title.

11. *The Babbage idea and its realization*, by Professor S. G. Hacker, State College of Washington.

Babbage's idea regarding his difference and analytical engines was discussed, with lantern slide illustrations of the respective parts completed by C. and H. P. Babbage. A realization of Babbage's "analytical engine" was first accomplished in the Harvard IBM-ASCC. The relative merits of the ASCC and ENIAC, as balanced machines, were cited. It was suggested that the average mathematics department should not allow interest in the high-speed automatic calculators to overshadow, in instruction and research, the importance of the modern "difference engines," electric and hand desk computers, and certain ledger-posting machines. The work in this connection of L. J. Comrie and others was cited. A brief critical bibliography of the latter, as well as of the Babbage machines and their modern counterparts, was given.

12. *Note on the solution of differential equations by symbolic methods*, by Professors W. H. Gage and R. D. James, University of British Columbia.

In textbooks on differential equations there is usually a fairly complete account of the use of symbolic methods in finding the particular integral of a linear differential equation with constant coefficients. The authors stated that it does not seem to have been pointed out, however, that a symbolic method may also be employed to find the complementary function. This paper described such a method which the authors have used to advantage in classes at the University of British Columbia.

13. *The adjoint of a partial differential equation*, by Professor L. H. McFarlan, University of Washington.

The formation of the adjoints of certain types of partial differential equations was shown to be simplified by use of Euler equations of the calculus of variations.

14. *Alternate digit cancellation*, by Mr. G. M. Peterson, University of British Columbia, introduced by the Secretary.

In certain cases the operation of cancelling alternate digits in numerator and denominator yields a result which coincides with ordinary cancellation (e.g. $16/64=1/4$). All two digit fractions having this property, for radix α, were determined.

15. *Some remarks on indices*, by Mr. G. E. Latta, University of British Columbia, introduced by Professor R. D. James.

The ξ of a prime to the base a, $\xi_a(p)$, is defined to be the exponent to which a belongs mod p, and if $a^{\alpha-1}+a^{\alpha-2}+\cdots+a+1$ is the smallest number of this form which p divides, then $\xi_a(p)=\alpha$. Certain interesting sequence of numbers such as the Fermat and Mersenne numbers are such that the ξ_a of all the factors of any one of these numbers is the same. This makes it possible to form a certain diophantine equation, the integral solutions of which lead to the factors of the number in question. By one of the properties of $\xi_a(n)$, the relative primeness of the Mersenne numbers was demonstrated, and a method was obtained for forming any desired number of infinite sequences of relatively prime integers.

16. *The graph of a function, a classroom note*, by Professor S. A. Jennings, University of British Columbia.

When functions of the form $f(x)$, $f(x, y)$, $f(x, y, z)$ are defined as numerical valued mappings of regions in one, two, or three dimensions, they are all of the type $P \to f(P)$. Methods of presenting this idea at an elementary level are discussed, and certain of its advantages for the teaching of multiple integration are pointed out. This paper was read by title.

S. G. HACKER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

Thirty-first Summer Meeting, Boulder, Colorado, September, 1949.

The following is a list of the Sections of the Association with dates of future meetings insofar as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, November 6, 1948
ILLINOIS, Peoria, May 13–14, 1949
INDIANA
IOWA, Drake University, Des Moines, April 15–16, 1949
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, Spring, 1949
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May, 1949
NORTHERN CALIFORNIA, San Francisco, January 29, 1949
OHIO, Ohio State University, Columbus, April 2, 1949
OKLAHOMA
PACIFIC NORTHWEST, Oregon State University, Corvallis, Spring, 1949
PHILADELPHIA, Philadelphia, November 27, 1948
ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949
SOUTHEASTERN, University of Alabama, University, March 18–19, 1949
SOUTHERN CALIFORNIA, John Muir Junior College, Pasadena, March 12, 1949
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, University of Buffalo, May, 1949
WISCONSIN, Lawrence College, Appleton, May, 1949

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